HYGROTHERMAL BUCKLING AND BLEACHING OF THIN PLATES

Project F020

Report 4

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By

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Executive Summary

A critical review of the theory of hygrothermal buckling of thin plates is presented. The preparation of this report is an extension of our previous report on buckling due to applied thrusts [1]. We are investigating the buckling behavior of thin plates because we believe that this science base is directly applicable to our study of cockle in paper. Most of the reviewed literature pertains to buckling arising from thermal stresses. In terms of mathematics, swelling due to moisture changes enters the formulation exactly in the same form as thermal expansion. Therefore, we can apply this directly to our focus of study on cockle in paper.

We present the equations governing the buckling of thin thermoelastic plates. Analyses of buckling for both rectangular and circular plates are reviewed. Both the case of small and large deflections are discussed. The effects of imperfections, thickness variations, and inelastic behavior are presented.

A common technique employed to obtain a solution, Berger's approximation, is discussed. For some cockle problems these may be a viable technique to implement.

In reviewing this literature on thermal buckling, we found several analyses that contained errors. These questionable results are discussed in the text, and create an opportunity for some original work of interest to the scientific community. We are now in the process of formulating the governing equations for our cockle model. This model will make use of the knowledge we gained in reviewing the literature reported herein.

I. GENERALIZED VON KARMAN EQUATIONS FOR ELASTIC ISOTROPIC AND ORTHOTROPIC PLATES SUBJECT TO HYGROEXPANSIVE OR THERMAL STRESS DISTRIBUTIONS

We begin our study of hygroexpansive and thermal buckling and bending of thin plates by deriving the generalized von Karman equations for elastic isotropic and orthotropic plates; the pertinent results will be presented in both rectilinear and polar coordinates for, respectively, rectangular plates and plates with a circular geometry. Further on in this report (i.e., in Chapter VII) the equations governing the bending and buckling behavior of thin plates either exhibiting viscoelastic (creep) behavior or undergoing plastic deformations will be presented.

A) Rectilinear Coordinates

We consider an isotropic thin plate of constant thickness h which occupies the domain Ω in the x, y plane ([1], Fig. II.11); as in [1] we employ the Kirchoff hypothesis, i.e., sections x =const., y =const. of the undeformed plate remain plane after deformation and also maintain their angle with respect to the deformed middle surface of the plate. In terms of the displacement components u, v, w of the middle surface of the plate we have the following generalization of equations (II.34) of [1] which applies when either hygroexpansive or thermal strains (or both) must be taken into account:

$$\begin{cases} \tilde{\epsilon} = \epsilon_{xx} - \epsilon_{HT} \\ \tilde{\epsilon}_{yy} = \epsilon_{yy} - \epsilon_{HT} \\ \tilde{\gamma}_{xy} = 2\tilde{\epsilon}_{xy} = 2\epsilon_{xy} \end{cases} \tag{I.1}$$

where $\epsilon_{xx}, \epsilon_{xy}, \epsilon_{yy}$ are given by (II.34) of [1], i.e.,

$$\begin{cases} \epsilon_{xx} &= u_{,x} + \frac{1}{2}w_{,x}^2 - \zeta w_{,xx} \\ \epsilon_{xy} &= \frac{1}{2}(u_{,y} + v_{,x} + w_{,x}w_{,y}) - \zeta w_{,xy} \\ \epsilon_{yy} &= v_{,y} + \frac{1}{2}w_{,y}^2 - \zeta w_{,yy} \end{cases}$$

with $-h \leq \zeta \leq h$ the (normal) distance from the middle surface of the plate, and

$$\epsilon_{HT} = \beta\delta H + \alpha\delta T \tag{I.2}$$

In (I.2), β is the (assumed constant) coefficient of hygroscopic expansion, α the thermal expansion coefficient, δH the change in moisture content and δT the change in temperature. For a static problem we have, in general

$$\begin{cases} \delta H = H(x,y,z) - H_o \\ \delta T = T(x,y,z) - T_o \end{cases} \tag{I.3}$$

with H_0, T_0, respectively, reference moisture and temperature levels. In writing down (I.1), (I.2) we have already assumed isotropy, i.e. $(\epsilon_{HT})_{xx} = (\epsilon_{HT})_{yy}$; for the case of rectilinear orthotropy we will have to introduce coefficients $\alpha_i, \beta_i, i = 1, 2$. We also note that for a purely hygroscopic problem $\delta T = 0$ while for an entirely thermal problem $\delta H = 0$.

Remarks: To simplify the presentation, and because almost all of the literature, to date, has dealt solely with problems of thermal buckling and bending, as opposed to hygroscopic buckling and bending (or a combination of both mechanisms) we will often write $\epsilon_{HT} \equiv \epsilon_T = \alpha\delta T$ or the natural generalization with respect to orthotropic response. However, in almost all the cases that will be discussed in this report $\alpha\delta T$ (in the isotropic case, for example) will be interchangeable with $\beta\delta H$.

For rectilinear isotropic response, the constitutive relations are given by [2]:

$$
\begin{cases}
\tilde{\epsilon}_{xx} &= \dfrac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) \\
\tilde{\epsilon}_{yy} &= \dfrac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}) \\
\tilde{\gamma}_{xy} &= \dfrac{2(1+\nu)}{E}\sigma_{xy}
\end{cases}
$$

or, in view of (I.1)

$$
\begin{cases}
\epsilon_{xx} - \epsilon_{HT} = \dfrac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) \\
\epsilon_{yy} - \epsilon_{HT} = \dfrac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}) \\
\epsilon_{xy} = (\dfrac{1+\nu}{E})\sigma_{xy}
\end{cases}
\tag{I.4}
$$

with ν the Poisson's ratio and E the Young's modulus. Alternatively, by solving (I.4) for the stress components, we have

$$
\begin{cases}
\sigma_{xx} &= \dfrac{E}{1-\nu^2}(\epsilon_{xx} + \nu\epsilon_{yy}) - \dfrac{E}{1-\nu} \cdot \epsilon_{HT} \\
\sigma_{yy} &= \dfrac{E}{1-\nu^2}(\epsilon_{yy} + \nu\epsilon_{xx}) - \dfrac{E}{1-\nu} \cdot \epsilon_{HT} \\
\sigma_{xy} &= \dfrac{E}{1+\nu} \cdot \epsilon_{xy} \equiv 2G\epsilon_{xy}
\end{cases}
\tag{I.5}
$$

where G is the shear modulus of the plate. With, e.g., $\epsilon_{HT} = \alpha\delta T \equiv \epsilon_T$, (I.4) says that as a consequence of a change in the heat content of the plate strains $\epsilon_{xx}, \epsilon_{xy}$, and ϵ_{yy} are caused by thermal expansion of the material comprising the plate as well as by stresses that may arise from applied loads or other sources. As in [1], the averaged stresses over the plate thickness h (assumed to be small) and the bending moments are given, respectively, by

$$\begin{cases} N_x &= \int_{-h/2}^{h/2} \sigma_{xx} dz \\ N_y &= \int_{-h/2}^{h/2} \sigma_{yy} dz \\ N_{xy} &= \int_{-h/2}^{h/2} \sigma_{xy} dz \end{cases} \tag{I.6a}$$

which are just equations (II.37) of [1], and

$$\begin{cases} M_x &= \int_{-h/2}^{h/2} \sigma_{xx} \cdot z dz \\ M_y &= \int_{-h/2}^{h/2} \sigma_{yy} \cdot z dz \\ M_{xy} &= \int_{-h/2}^{h/2} \sigma_{xy} \cdot z dz \end{cases} \tag{I.6b}$$

In view of the definitions of ϵ_{xx}, ϵ_{xy}, and ϵ_{yy}, if we define, in the usual way, the middle surface strains by

$$\begin{cases} \epsilon_{xx}^o &= u_{,x} + \frac{1}{2} w_{,x}^2 \\ \epsilon_{yy}^o &= v_{,y} + \frac{1}{2} w_{,y}^2 \\ \epsilon_{xy}^o &= \frac{1}{2}(u_{,y} + v_{,x} + w_{,x} w_{,y}) \end{cases} \tag{I.7}$$

so that

$$\begin{cases} \epsilon_{xx} &= \epsilon_{xx}^o - \zeta w_{,xx} \\ \epsilon_{xy} &= \epsilon_{xy}^o - \zeta w_{,xy} \\ \epsilon_{yy} &= \epsilon_{yy}^o - \zeta w_{,yy} \end{cases} \tag{I.8}$$

then, by virtue of (I.6a), (I.5), and (I.8)

$$\begin{cases} N_x &= \frac{Eh}{1-\nu^2}(\epsilon_{xx}^o + \nu \epsilon_{yy}^o) - N_{HT} \\ N_y &= \frac{Eh}{1-\nu^2}(\epsilon_{yy}^o + \nu \epsilon_{xx}^o) - N_{HT} \\ N_{xy} &= 2Gh\epsilon_{xy}^o \end{cases} \tag{I.9}$$

where

$$N_{HT} = \frac{E}{1-\nu} \cdot \int_{-h/2}^{h/2} \epsilon_{HT} dz \tag{I.10}$$

For the purely thermal situation in which $\epsilon_{TH} \equiv \epsilon_T = \alpha \triangle T(x, y, z)$

$$N_{HT} = N^T = \frac{\alpha E}{1-\nu} \int_{-h/2}^{h/2} \delta T(x, y, z) dz \tag{I.11a}$$

while in the entirely hygroscopic case with $\epsilon_{TH} = \epsilon_H$

$$N_{HT} = N^H = \frac{\beta E}{1-\nu} \int_{-h/2}^{h/2} \delta H(x, y, z) dz \tag{I.11b}$$

Equations (I.9) may be found, e.g., in §9.4 of [3], with $N_{HT} = N^T$.

In an analogous fashion, we may compute that, by virtue of (I.6b), (I.5), and (I.8), the bending moments are given by

$$\begin{cases} M_x &= -K(w_{,xx} + \nu w_{,yy}) - M_{HT} \\ M_y &= -K(w_{,yy} + \nu w_{,xx}) - M_{HT} \\ M_{xy} &= -(1-\nu) K w_{,xy} \end{cases} \tag{I.12}$$

where $K = \dfrac{Eh^2}{12(1-\nu^2)}$ is the usual plate stiffness for the isotropic case while the hygrothermal moment M_{HT} is given by

$$M_{HT} = \frac{E}{1-\nu} \int_{-h/2}^{h/2} \epsilon_{HT} z dz \tag{I.13}$$

For $\epsilon_{HT} \equiv \epsilon_T$,

$$M_{HT} = M^T = \frac{\alpha E}{1-\nu} \cdot \int_{-h/2}^{h/2} \delta T(x, y, z) z dz \tag{I.14}$$

while for $\epsilon_{HT} = \epsilon_H$,

$$M_{HT} = M^H = \frac{\beta E}{1-\nu} \cdot \int_{-h/2}^{h/2} \delta H(x, y, z) z dz \tag{I.15}$$

The relations (I.12), with $M_{HT} = M^T$ may also be found, e.g., in §9.4 of [3].

Remarks: In certain situations it may be the case that the coefficients α and/or β are field-dependent, i.e., $\alpha = \alpha(\delta T), \beta = \beta(\delta H)$. In such case one would have, e.g.,

$$\begin{cases} N^T = \dfrac{E}{1-\nu} \displaystyle\int_{-h/2}^{h/2} \alpha(\delta T) \cdot \delta T dz \\ M^T = \dfrac{E}{1-\nu} \displaystyle\int_{-h/2}^{h/2} \alpha(\delta T) \cdot \delta T z dz \end{cases} \tag{I.16}$$

with analogous expressions for N^H, M^H.

The equilibrium equations which apply in the present situation (see Figs. II.8, II.9 of [1]) are precisely the same relations which hold in the absence of hygrothermal strains, i.e., (II.42), (II.43), and (II.44) of [1]; we write these (in the absence of an initial deflection and a distributed normal loading) in the form

$$\begin{cases} N_{x,x} + N_{xy,y} = 0 \\ N_{xy,x} + N_{y,y} = 0 \end{cases} \tag{I.17a}$$

$$Q_{xz,x} + Q_{yz,y} + N_x w_{,xx} + N_y w_{,yy} + 2N_{xy} w_{,xy} = 0 \tag{I.17b}$$

$$\begin{cases} M_{xy,x} + M_{y,y} - Q_{yz} = 0 \\ M_{yx,y} + M_{x,x} - Q_{xz} = 0 \end{cases} \quad (M_{xy} = M_{yx}) \tag{I.17c}$$

Eliminating Q_{yz} and Q_{xz} from among the equations in (I.17b, c) we obtain

$$M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy} = 0 \tag{I.18}$$

Substituting for the moments M_x, M_y, and M_{xy} in (I.18) then yields

$$K\triangle^2 w = N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy} - \triangle M_{HT} \tag{I.19}$$

where $\triangle = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ is the two-dimensional Laplacian while $\triangle^2$ is the biharmonic operator. Modifications (which will be discussed later in this report) must be made to (I.19) if imperfection buckling is considered, i.e., it the plate possesses an initial prebuckling deflection

$w_0 = w_0(x,y)$ or is subject to a transverse normal loading. As in [1] we may introduce the Airy stress function $\Phi(x,y)$ by

$$N_x = \Phi_{xyy}, N_y = \Phi_{,xx}, N_{xy} = -\Phi_{,xy} \tag{I.20}$$

in which case equations (I.17a) are satisfied identically while (I.19) becomes

$$K\triangle^2 w = \Phi_{,yy}w_{,xx} - 2\Phi_{,xy}w_{,xy} + \Phi_{,xx}w_{,yy} - \triangle M_{HT} \tag{I.21}$$

From the compatibility equation

$$\frac{\partial^2}{\partial y^2}\epsilon^0_{xx} + \frac{\partial^2}{\partial x^2}\epsilon^0_{yy} - 2\frac{\partial^2 \epsilon^0_{xy}}{\partial x \partial y} = (\frac{\partial^2 w}{\partial x \partial y})^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}, \tag{I.22}$$

where $\epsilon^0_{xx}, \epsilon^0_{yy}, \epsilon^0_{xy}$ are the middle surface strains, as given by (I.7) and the constitutive relations (I.9), we easily obtain the second of the two generalized von Karman equations which apply in the case of hygrothermal buckling, i.e.,

$$\triangle^2\Phi = Eh\{(\frac{\partial^2 w}{\partial x \partial y})^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\} - (1-\nu)\triangle N_{HT} \tag{I.23}$$

As in [1] we may introduce the nonlinear (bracket) differential operator by

$$[f,g] = f_{,yy}g_{,xx} - 2f_{,xy}g_{,xy} + f_{,xx}g_{,yy}$$

and write (I.23) in the form

$$\triangle^2\Phi = -\frac{1}{2}Eh[w,w] - (1-\nu)\triangle N_{HT} \tag{I.24}$$

With $\triangle M_{HT} = \triangle N_{HT} = 0$, the system (I.21), (I.24) reduces to the standard system of von Karman equations which apply in the case of linear, isotropic, elastic response in rectilinear Cartesian coordinates in the absence of both an initial prebuckling deflection and an applied transverse normal loading.

For the case in which the thin plate exhibits linear elastic behavior, but possesses rectilinear orthotropic symmetry, the generalized von Karman equations governing hygrothermal buckling may be derived as follows: for a constant thickness orthotropic think plate, in which the x and y axes coincide with the principle directions, the constitutive equations have the form

$$\left\{ \begin{array}{c} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{array} \right\} = \left[\begin{array}{ccc} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \\ 0 & 0 & c_{66} \end{array} \right] \left\{ \begin{array}{c} \epsilon_{xx} - \epsilon_{HT}^{1} \\ \epsilon_{yy} - \epsilon_{HT}^{2} \\ \gamma_{xy} \end{array} \right\} \tag{I.25}$$

where the hygrothermal strains have the form

$$\left\{ \begin{array}{rcl} \epsilon_{HT}^{1} & = & \beta_1 \delta H + \alpha_1 \delta T \\ \epsilon_{HT}^{2} & = & \beta_2 \delta H + \alpha_2 \delta T \end{array} \right. \tag{I.26}$$

with α_1, α_2 the coefficients of linear thermal expansion along the x and y axes, respectively, and β_1, β_2 the coefficients of hygroscopic expansion. For now, we shall assume that the α_i and β_i are constant, $i = 1, 2$. In (I.25) the elastic constants are given by the following expressions in which $E_1, E_2, \nu_{12}, \nu_{21}$, and G_{12}, respectively, are the Young's moduli, Poisson's ratios, and shear modulus associated with the principal directions:

$$\left\{ \begin{array}{l} c_{11} = E_1/(1 - \nu_{12}\nu_{21}) \\ c_{12} = E_2\nu_{21}/(1 - \nu_{12}\nu_{21}) \\ c_{21} = E_1\nu_{12}/(1 - \nu_{12}\nu_{21}) \\ c_{22} = E_2/(1 - \nu_{12}\nu_{21}) \\ c_{66} = G_{12} \end{array} \right. \tag{I.27}$$

Also, $E_1\nu_{12} = E_2\nu_{21}$ so that $c_{12} = c_{21}$. As in [1], the constants $D_{ij} = c_{ij}h^3/12$ are the associated rigidities (i.e., stiffness ratios) of the orthotropic plate. Employing (I.27) we have for the bending rigidities about the x and y axes, respectively,

$$D_{11} = \frac{E_1 h^3}{12(1 - \nu_{12}\nu_{21})} \text{ and } D_{22} = \frac{E_2 h^3}{12(1 - \nu_{12}\nu_{21})} \tag{I.28a}$$

while

$$D_{66} = \frac{G_{12} h^3}{12} \tag{I.28b}$$

is the twisting rigidity. Often, the ratios D_{12}/D_{22} and D_{26}/D_{11} are termed reduced Poisson's ratios.

We write (I.25) out in the form

$$\begin{cases} \sigma_{xx} = c_{11}(\epsilon^0_{xx} - \zeta w_{,xx}) + c_{12}(\epsilon^0_{yy} - \zeta w_{,yy}) \\ \qquad\qquad -c_{11}\epsilon^1_{HT} - c_{12}\epsilon^2_{HT} \\ \sigma_{yy} = c_{21}(\epsilon^0_{xx} - \zeta w_{,xx}) + c_{22}(\epsilon^0_{yy} - \zeta w_{,yy}) \\ \qquad\qquad -c_{21}\epsilon^1_{HT} - c_{22}\epsilon^2_{HT} \\ \sigma_{xy} = 2c_{66}(\epsilon^0_{xy} - \zeta w_{,xy}) \end{cases} \tag{I.29}$$

where the middle surface strains are given by (I.7).

The averaged stresses (over the plate thickness) are still given by (I.6a) so that, by virtue of (I.29), and (I.27)

$$\begin{cases} N_x = \left\{\dfrac{E_1 h}{1-\nu_{12}\nu_{21}}\right\}\epsilon^0_{xx} + \left\{\dfrac{E_2\nu_{21}h}{1-\nu_{12}\nu_{21}}\right\}\epsilon^0_{yy} \\ \qquad\qquad -(N^{11}_{HT} + N^{12}_{HT}) \\ N_y = \left\{\dfrac{E_1\nu_{12}h}{1-\nu_{12}\nu_{21}}\right\}\epsilon^0_{xx} + \left\{\dfrac{E_2 h}{1-\nu_{12}\nu_{21}}\right\}\epsilon^0_{yy} \\ \qquad\qquad -N^{21}_{HT} + N^{22}_{HT}) \\ N_{xy} = (2G_{12}h)\epsilon^0_{xy} \end{cases} \tag{I.30}$$

where

$$\begin{cases} N^{11}_{HT} = \left(\dfrac{E_1}{1-\nu_{12}\nu_{21}}\right)\displaystyle\int_{-h/2}^{h/2}\epsilon^1_{HT}dz \\ N^{12}_{HT} = \left(\dfrac{E_2\nu_{21}}{1-\nu_{12}\nu_{21}}\right)\displaystyle\int_{-h/2}^{h/2}\epsilon^2_{HT}dz \\ N^{21}_{HT} = \left(\dfrac{E_1\nu_{12}}{1-\nu_{12}\nu_{21}}\right)\displaystyle\int_{-h/2}^{h/2}\epsilon^1_{HT}dz \\ N^{22}_{HT} = \left(\dfrac{E_2}{1-\nu_{12}\nu_{21}}\right)\displaystyle\int_{-h/2}^{h/2}\epsilon^2_{HT}dz \end{cases} \tag{I.31}$$

Remarks: In the special case of rectilinear isotropic elastic response (I.30), (I.31) reduce as follows: we set $E_1 = E_2 = E$, $\nu_{12} = \nu_{21} = \nu$, and $G_{12} = G$. Also

$$\epsilon_{HT}^1 = \epsilon_{HT}^2 \equiv \beta\delta H + \alpha\delta T \equiv \epsilon_{HT}$$

as $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$. Then

$$\begin{cases} N_{HT}^{11} + N_{HT}^{12} = \dfrac{E}{1-\nu}\displaystyle\int_{-h/2}^{h/2} \epsilon_{HT} dz \\ N_{HT}^{21} + N_{HT}^{22} = \dfrac{E}{1-\nu}\displaystyle\int_{-h/2}^{h/2} \epsilon_{HT} dz \end{cases} \tag{I.32}$$

and it is clear that (I.30) reduces to (I.9) with $N_{HT} = N_{HT}^{11} + N_{HT}^{12} = N_{HT}^{21} + N_{HT}^{22}$ given by (I.10).

From (I.30) we have, immediately, that

$$\epsilon_{xy}^0 = \left(\frac{1}{2G_{12}h}\right) N_{xy} \tag{I.33a}$$

while the linear algebraic system

$$\begin{cases} c_{11}\epsilon_{xx}^0 + c_{12}\epsilon_{yy}^0 = \dfrac{1}{h}(N_x + \tilde{N}_{HT}^1) \\ c_{21}\epsilon_{xx}^0 + c_{22}\epsilon_{yy}^0 = \dfrac{1}{h}(N_y + \tilde{N}_{HT}^2) \end{cases}$$

with

$$\begin{cases} \tilde{N}_{HT}^1 = N_{HT}^{11} + N_{HT}^{12} \\ \tilde{N}_{HT}^2 = N_{HT}^{21} + N_{HT}^{22} \end{cases}$$

yields

$$\epsilon_{xx}^0 = \frac{1}{|c_{ij}|h}\left\{\left(\frac{E_2}{1-\nu_{12}\nu_{21}}\right)(N_x + \tilde{N}_{HT}^1) - \left(\frac{E_2\nu_{21}}{1-\nu_{12}\nu_{21}}\right)(N_y + \tilde{N}_{HT}^2)\right\} \tag{I.33b}$$

$$\epsilon_{yy}^0 = \frac{1}{|c_{ij}|h}\left\{\left(\frac{E_1}{1-\nu_{12}\nu_{21}}\right)(N_y + \tilde{N}_{HT}^2) - \left(\frac{E_1\nu_{12}}{1-\nu_{12}\nu_{21}}\right)(N_x + \tilde{N}_{HT}^1)\right\} \tag{I.33c}$$

with

$$|c_{ij}| = \begin{vmatrix} \dfrac{E_1}{1-\nu_{12}\nu_{21}} & \dfrac{E_2\nu_{21}}{1-\nu_{12}\nu_{21}} \\ \dfrac{E_1\nu_{12}}{1-\nu_{12}\nu_{21}} & \dfrac{E_2}{1-\nu_{12}\nu_{21}} \end{vmatrix},$$

i.e.

$$|c_{ij}| = E_1E_2/(1 - \nu_{12}\nu_{21}) \tag{I.34}$$

Employing (I.34) in (I.33b,c) and recalling (I.33a) we easily find that the inverted constitutive relations assume the form

$$\begin{cases} \epsilon^0_{xx} = \dfrac{1}{hE_1}\left\{(N_x + \tilde{N}^1_{HT}) - \nu_{21}(N_y + \tilde{N}^2_{HT})\right\} \\ \epsilon^0_{yy} = \dfrac{1}{hE_2}\left\{(N_y + \tilde{N}^2_{HT}) - \nu_{12}(N_x + \tilde{N}^1_{HT})\right\} \\ \epsilon^0_{xy} = \dfrac{1}{2hG_{12}}N_{xy} \end{cases} \tag{I.35}$$

To compute the bending moments M_x, M_y, and M_{xy} we employ the constitutive relations (I.29) in (I.6b) so as find

$$\begin{cases} M_x = -\dfrac{h^3}{12}(c_{11}w_{,xx} + c_{12}w_{,yy}) - (M^{11}_{HT} + M^{12}_{HT}) \\ M_y = -\dfrac{h^3}{12}(c_{21}w_{,xx} + c_{22}w_{,yy}) - (M^{21}_{HT} + M^{22}_{HT}) \\ M_{xy} = -\dfrac{h^3}{6}c_{66}w_{,xy} \end{cases}$$

where the hygrothermal moments M^{ij}_{HT} are given by

$$\begin{cases} M^{11}_{HT} = c_{11}\displaystyle\int_{-h/2}^{h/2} \epsilon^1_{HT}(x,y,z)zdz \\ M^{12}_{HT} = c_{12}\displaystyle\int_{-h/2}^{h/2} \epsilon^2_{HT}(x,y,z)zdz \\ M^{21}_{HT} = c_{21}\displaystyle\int_{-h/2}^{h/2} \epsilon^1_{HT}(x,y,z)zdz \\ M^{22}_{HT} = c_{22}\displaystyle\int_{-h/2}^{h/2} \epsilon^2_{HT}(x,y,z)zdz \end{cases} \tag{I.36}$$

Employing the rigidities $D_{ij} = c_{ij}h^3/12$, the bending moments may be written in the form

$$\begin{cases} M_x = -(D_{11}w_{,xx} + D_{12}w_{,yy}) - (M^{11}_{HT} + M^{12}_{HT}) \\ M_y = -(D_{21}w_{,xx} + D_{22}w_{,yy}) - (M^{21}_{HT} + M^{22}_{HT}) \\ M_{xy} = -2D_{66}w_{,xy} \end{cases} \tag{I.37}$$

Equations (I.36), (I.37) generalize the relations (II.67a, b, c) of [1] and reduce to the latter expressions for the bending moments when $\alpha_i = \beta_i = 0, i = 1, 2$.

For the orthotropic case, the equilibrium equations (I.17a), (I.18) still apply with the averaged stresses given by (I.30), (I.31) and the bending moments by (I.36), (I.37). As in the (rectilinear) isotropic case, we introduce the Airy function Φ which is defined by (I.20) and the pair of equations (I.17a) is satisfied identically. Substituting into (I.18), from (I.20) and (I.37), we next find that (I.18) implies that

$$D_{11}w_{,xxxx} + \{D_{12} + 4D_{66} + D_{21}\}w_{,xxyy} + D_{22}w_{,yyyy} = [\Phi, w] - \{(\tilde{M}^1_{HT})_{,xx} + (\tilde{M}^2_{HT})_{,yy}\} \quad \text{(I.38)}$$

where

$$\begin{cases} \tilde{M}^1_{HT} &= M^{11}_{HT} + M^{12}_{HT} \\ \\ \tilde{M}^2_{HT} &= M^{21}_{HT} + M^{22}_{HT} \end{cases} \quad \text{(I.39)}$$

With $\alpha_i = \beta_i = 0, i = 1, 2$, (I.38) reduces to the first of the von Karman equations for the non-hygrothermal case, e.g., (II.68) of [1].

Remarks: Various special cases of (I.38) may be written down by, e.g., taking the $\beta_i = 0$ in (I.26), assuming the α_i are constants, and taking specific forms for $\delta T(x, y, z)$. Such considerations will be relegated to the discussion, later on in this report, of (mostly thermal) bending and buckling analyses which have appeared in the literature.

The second generalized von Karman equation for hygrothermal bending and bucking, for the case of rectilinear orthotropic symmetry, follows as in the isotropic case from the compatability equation (I.22), i.e.,

$$\frac{\partial^2}{\partial y^2}\epsilon^o_{xx} + \frac{\partial^2}{\partial x^2}\epsilon^o_{yy} - 2\frac{\partial^2 \epsilon^o_{xy}}{\partial x \partial y} = -\frac{1}{2}[w, w] \quad \text{(I.40)}$$

In the present situation the middle surface strains $\epsilon^o_{xx}, \epsilon^o_{xy}$, and ϵ^o_{yy} are given by (I.35); introducing the Airy function into (I.35), we now write this system in the form

$$\begin{cases} \epsilon^o_{xx} &= \dfrac{1}{hE_1}\cdot(\Phi_{,yy}-\nu_{21}\Phi_{,xx})+\dfrac{1}{hE_1}(\tilde{N}^1_{HT}-\nu_{21}\tilde{N}^2_{HT}) \\ \epsilon^o_{yy} &= \dfrac{1}{hE_2}\cdot(\Phi_{,xx}-\nu_{12}\Phi_{,yy})+\dfrac{1}{hE_2}(\tilde{N}^2_{HT}-\nu_{12}\tilde{N}^1_{HT}) \\ \epsilon^o_{xy} &= -\dfrac{1}{2hG_{12}}\Phi_{,xy} \end{cases} \tag{I.41}$$

and then substitute into (I.40) so as to obtain

$$\begin{aligned} &\frac{1}{E_1h}\Phi_{,yyyy}+\frac{1}{h}(\frac{1}{G_{12}}-\frac{2\nu_{12}}{E_2})\Phi_{,xxyy} \\ &+\frac{1}{E_2h}\Phi_{,xxxx}=-\frac{1}{2}[w,w] \\ &-\frac{1}{E_1h}(\tilde{N}^1_{HT}-\nu_{21}\tilde{N}^2_{HT})_{,yy}-\frac{1}{E_2h}(\tilde{N}^2_{HT}-\nu_{12}\tilde{N}^1_{HT})_{,xx} \end{aligned} \tag{I.42}$$

Equation (I.42) reduces to (II.69) of [1] for the non-hygrothermal case when $\alpha_i = \beta_i = 0, i = 1,2$, in which case $\tilde{N}^2_{HT} = \tilde{N}^2_{HT} = 0$. Various special cases of (I.42) will appear later on in this chapter when we consider specific (mostly, thermal buckling and bending) problems that have appeared in the literature. For a thin linear elastic plate, which exhibits rectilinear orthotropic symmetry, the complete system of generalized von Karman equations, incorporating hygrothermal expansion and contraction, consists of (I.38) and (I.42); these equations hold in the absence of initial deflections and a transverse (normal) loading. Systems of equations which are similar to (but less general than) (I.38), (I.42) for the case of thermal buckling and bending of thin, linearly elastic, rectilinearly orthotropic plates have appeared in several places in the literature, e.g., [4], [5], and [6], as well as in §7.2 and §9.2 of [3].

B) Polar Coordinates

In this subsection we will present versions of the generalized von Karman equations in polar coordinates for thin linearly elastic thin plates exhibiting isotropic response as well as cylindrically orthotropic response. We will also indicate how the generalized (hygrothermal) form of von Karman's equations may be obtained, in polar coordinates, for thin linearly elastic plates exhibiting rectilinear orthotropic symmetry.

In a cylindrical coordinate system (r, θ, z), with the polar coordinates (r, θ) describing the middle surface of the plate, the expression in (I.2) for the hygrothermal strain remains, essentially, invariant except that $\S H$ and $\S T$ are now functions of (r, θ, z), i.e.,

$$e^*_{HT} = \beta\delta H(r, \theta, z) + \alpha\delta T(r, \theta, z) \tag{I.43}$$

In lieu of (I.1) we have

$$\left\{ \begin{array}{lcl} \tilde{e}_{rr} & = & e_{rr} - e^*_{HT} \\ \tilde{e}_{\theta\theta} & = & e_{\theta\theta} - e^*_{HT} \\ \tilde{\gamma}_{r\theta} & = & \gamma_{r\theta} \end{array} \right. \tag{I.44}$$

where $e_{rr}, e_{\theta\theta}$, and $\gamma_{r\theta}$ are given in terms of the displacement components u_r, u_θ in the middle surface of the plate and the out of plane displacement $w = w(r, \theta)$ by the relations in (II.71) of [1], i.e.,

$$\left\{ \begin{array}{lcl} e_{rr} & = & \dfrac{\partial u_r}{\partial r} + \dfrac{1}{2}(\dfrac{\partial w}{\partial r})^2 - \zeta\dfrac{\partial^2 w}{\partial r^2} \\ e_{\theta\theta} & = & \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{1}{\partial r^2}(\dfrac{\partial w}{\partial \theta})^2 - \zeta(\dfrac{1}{r}\dfrac{\partial w}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial \theta^2}) \\ \gamma_{r\theta} & = & \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} + \dfrac{1}{r}(\dfrac{\partial u_r}{\partial \theta}) + \dfrac{1}{r}(\dfrac{\partial w}{\partial r})(\dfrac{\partial w}{\partial \theta}) - 2\zeta(\dfrac{1}{r}\dfrac{\partial^2 w}{\partial r\partial \theta} - \dfrac{1}{r^2}\dfrac{\partial w}{\partial \theta}) \end{array} \right.$$

Middle surface strains $e^o_{rr}, e^o_{\theta\theta}, \gamma^o_{r\theta}$ are obtained from these relations by simply setting $\zeta = 0$.

The stress components $\sigma_{rr}, \sigma_{\theta\theta}$, and $\sigma_{r\theta}$ must satisfy the equilibrium equations delineated, e.g., in (II.72a, b) of [1]; they are related to the stress components in rectangular Cartesian coordinates by the equations in (II.73) of [1], i.e.,

$$\begin{aligned}\sigma_{rr} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta \\ \sigma_{\theta\theta} &= \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta \\ \sigma_{r\theta} &= (\sigma_{yy} - \sigma_{xx})\sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta)\end{aligned}$$

Assuming that the thin plate exhibits linearly elastic response, we have in lieu of (I.4), the constitutive relations

$$\begin{cases} e_{rr} - e^*_{HT} = \dfrac{1}{E}(\sigma_{rr} - \nu\sigma_{\theta\theta}) \\ e_{\theta\theta} - e^*_{HT} = \dfrac{1}{E}(\sigma_{\theta\theta} - \nu\sigma_{rr}) \\ e_{r\theta} = (\dfrac{1+\nu}{E})\sigma_{r\theta} \end{cases} \tag{I.45}$$

whose inverted form is

$$\begin{cases} \sigma_{rr} = \dfrac{E}{1-\nu^2}(e_{rr} + \nu e_{\theta\theta}) - \dfrac{E}{1-\nu}e^*_{HT} \\ \sigma_{\theta\theta} = \dfrac{E}{1-\nu^2}(e_{\theta\theta} + \nu e_{rr}) - \dfrac{E}{1-\nu}e^*_{HT} \\ \sigma_{r\theta} = (\dfrac{E}{1+\nu})e_{r\theta} \equiv G\gamma_{r\theta} \end{cases} \tag{I.46}$$

The averaged stresses and bending moments in polar coordinates are defined in the obvious way as the natural counterparts of (I.6a, b), i.e.

$$\begin{cases} N_r = \displaystyle\int_{-h/2}^{h/2} \sigma_{rr}dz \\ M_r = \displaystyle\int_{-h/2}^{h/2} \sigma_{rr}zdz \end{cases} \quad (etc.)$$

So that, by virtue of (I.46)

$$\begin{cases} N_r = \dfrac{Eh}{1-\nu^2}(e^0_{rr} + \nu e^o_{\theta\theta}) - N^*_{HT} \\ N_\theta = \dfrac{Eh}{1-\nu^2}(e^0_{\theta\theta} + \nu e^0_{rr}) - N^*_{HT} \\ N_{r\theta} = 2Ghe^0_{r\theta} \end{cases} \tag{I.47}$$

where $N_{HT}^* = \dfrac{E}{1-\nu}\displaystyle\int_{-h/2}^{h/2} e_{HT}^* dz$.

It is easily shown that the polar coordinate equivalent of the equilibrium equations (I.17a), i.e.,

$$\begin{cases} N_{r,r} + \dfrac{1}{r}N_{r\theta,\theta} + \dfrac{1}{r}(N_r - N_\theta) = 0 \\ N_{r\theta,r} + \dfrac{1}{r}N_{\theta,\theta} + \dfrac{2}{r}N_{r\theta} = 0 \end{cases} \tag{I.48}$$

is satisfied by introducing the Airy (stress) function $\Phi = \Phi(r,\theta)$ defined by

$$\begin{cases} N_r &= \dfrac{1}{r}\Phi_{,r} + \dfrac{1}{r^2}\Phi_{,\theta\theta} \\ N_\theta &= \Phi_{,rr} \\ N_{r\theta} &= \dfrac{1}{r^2}\Phi_{,\theta} - \dfrac{1}{r}\Phi_{,r\theta} \end{cases} \tag{I.49}$$

For the isotropic situation under consideration, the generalized von Karman equations (in polar coordinates) can now be derived by simply employing the polar coordinate equivalent forms of the equilibrium equation (I.18) and the compatability equation (I.22); specifically, in the polar coordinate equivalent form of (I.27) one would substitute, from (I.47) and (I.49), for the middle surface strains $e_{rr}^0, e_{\theta\theta}^0$, and $e_{r\theta}^0$ in terms of radial and angular derivatives of $\Phi(r,\theta)$, while in the polar coordinate equivalent form of (I.18) we would substitute for radial and angular derivatives of the bending moments M_r, M_θ, and $M_{r\theta}$ the equivalent expressions in terms of derivatives of $w = w(r,\theta)$. A more direct way to obtain the generalized von Karman system, in polar coordinates, for linear isotropic response, is to note that the differential operators present in the system (I.21), (I.24) are invariant with respect to linear transformations of the coordinate system; in particular we have, in lieu of (I.21), (I.24), the system

$$\begin{cases} K\triangle^2 w(r,\theta) &= [\Phi(r,\theta), w(r,\theta)] - \triangle M_{HT}^* \\ \triangle^2\Phi(r,\theta) &= -\dfrac{1}{2}Eh[w(r,\theta), w(r,\theta)] - (1-\nu)\triangle N_{HT}^* \end{cases} \tag{I.50}$$

where

$$\begin{cases} N^*_{HT} &= \dfrac{E}{1-\nu}\displaystyle\int_{-h/2}^{h/2} e^*_{HT} dz \\ M^*_{HT} &= \dfrac{E}{1-\nu}\displaystyle\int_{-h/2}^{h/2} e^*_{HT} z dz \end{cases} \tag{I.51}$$

while $\triangle^2 w$ is given by (II.75a) of [1], i.e.,

$$\begin{aligned} \triangle^2 w &= w_{,rrrr} + \frac{2}{r} w_{,rrr} - \frac{1}{r^2} w_{,rr} \\ &+ \frac{2}{r^2} w_{,rr\theta\theta} + \frac{1}{r^3} w_{,r} - \frac{2}{r^3} w_{,r\theta\theta} \\ &+ \frac{1}{r^4} w_{,\theta\theta\theta\theta} + \frac{4}{r^4} w_{,\theta\theta}, \end{aligned}$$

with an analogous expression for $\triangle^2\Phi$, while

$$\begin{aligned} [\Phi, w] &= w_{,rr}(\frac{1}{r}\Phi_{,r} + \frac{1}{r^2}\Phi_{,\theta\theta}) \\ &+ (\frac{1}{r} w_{,r} + \frac{1}{r^2} w_{,\theta\theta})\Phi_{,rr} \\ &- 2(\frac{1}{r} w_{,r\theta} - \frac{1}{r^2} w_{,\theta})(\frac{1}{r}\Phi_{,r\theta} - \frac{1}{r^2}\Phi_{,\theta}), \\ &= N_r w_{,rr} - 2N_{r\theta}(\frac{1}{r^2} w_{,\theta} - \frac{1}{r} w_{,r\theta}) \\ &+ N_\theta(\frac{1}{r} w_{,r} + \frac{1}{r^2} w_{,\theta\theta}) \\ [w, w] &= 2\{w_{,rr}(\frac{1}{r} w_{,r} + \frac{1}{r^2} w_{,\theta\theta}) \\ &- (\frac{1}{r} w_{,r\theta} - \frac{1}{r^2} w_{,\theta})^2\} \end{aligned}$$

and

$$\begin{aligned} \triangle M^*_{HT}(r,\theta) &= (M^*_{HT})_{,rr} + \frac{1}{r}(M^*_{HT})_{,r} \\ &+ \frac{1}{r^2}(M^*_{HT})_{,\theta\theta} \end{aligned} \tag{I.52}$$

with an analogous expression for $\triangle N^*_{HT}(r,\theta)$. If $\delta H_{,\theta} = \delta T_{,\theta} = 0$ so that

$$e^*_{HT}(r,z) = \beta\delta H(r,z) + \alpha\delta T(r,z), \tag{I.53}$$

and we consider only radial symmetric deformations of the plate, then the generalized von Karman system for the case of linearly elastic, isotropic response reduces to the pair of equations

$$K[w'''' + \frac{2}{r}w''' - \frac{1}{r}w'] \\ = N_r w'' + \frac{N_\theta}{r}w' - \{M^{*''}_{HT} + \frac{1}{r}M^{*'}_{HT}\}, \tag{I.54a}$$

where $' = \frac{d}{dr}, N_r = \frac{1}{r}\Phi', N_\theta = \Phi'' , M^*_{HT}(r) = \frac{E}{1-\nu}\int_{-h/2}^{h/2} e^*_{HT}(r,z)zdz,$

and

$$\frac{1}{Eh}[\Phi'''' + \frac{2}{r}\Phi''' - \frac{1}{r^2}\Phi'' + \frac{1}{r^3}\Phi'] \\ = -\frac{1}{r}w'w'' - \frac{(1-\nu)}{Eh}\{N^{*''}_{HT} + \frac{1}{r}N^{*'}_{HT}\} \tag{I.54b}$$

Both the systems (I.50) and (I.54a, b) neglect the effects of initial (prebuckling) deflections and an applied transverse loading; special cases of these systems, which have appeared in the literature in connection with problems associated with the thermal bucking of isotropic, linearly elastic circular plates will be analyzed later in this report.

For a linearly elastic, orthotropic body exhibiting cylindrical orthotropy there exist three planes of elastic symmetry; one of these is normal to the plane of anisotropy, the second passes through that axis, and the third is orthogonal to the first two. For a thin plate the first plane of elastic symmetry, for a cylindrically orthotropic material, is chosen parallel to the middle plane of the plate; with this convention, the constitutive equations generalize those in (I.45) for an isotropic plate (in polar coordinates) and assume the form

$$\left\{\begin{array}{rcl} e_{rr} - e^r_{HT} & = & \frac{1}{E_r}\sigma_{rr} - \frac{\nu_\theta}{E_\theta}\sigma_{\theta\theta} \\ e_{\theta\theta} - e^\theta_{HT} & = & -\frac{\nu_r}{E_r}\sigma_{rr} + \frac{1}{E_\theta}\sigma_{\theta\theta} \\ \gamma_{r\theta} & = & \frac{1}{G_{r\theta}}\sigma_{r\theta} \end{array}\right. \tag{I.55}$$

where the radial and angular hygrothermal strains e^r_{HT} and e^θ_{HT} are defined, respectively, by

$$\begin{cases} e^r_{HT} &= \beta_r \delta H(r,\theta,z) + \alpha_r \delta T(r,\theta,z) \\ e^\theta_{HT} &= \beta_\theta \delta H(r,\theta,z) + \alpha_\theta \delta T(r,\theta,z) \end{cases} \tag{I.56}$$

with α_r, α_θ the coefficients of thermal expansion in the radial and angular directions, respectively, and β_r, β_θ the coefficients of hygroexpansion in the radial and angular directions, respectively. Also, in (I.55), E_r and E_θ are the Young's moduli for tension (or compression) in the radial and tangential directions, respectively, while ν_r and ν_θ are the corresponding principal Poisson's ratios and $G_{r\theta}$ is the shear modulus which characterizes the change of angle between the radial and angular directions. We note that for a cylindrically (or polar) orthotropic body we have $E_r\nu_\theta = E_\theta\nu_r$ so that the constitutive equations (I.55) may be rewritten in the form

$$\begin{cases} e_{rr} &= \frac{1}{E_r}(\sigma_{rr} - \nu_r \sigma_{\theta\theta}) + e^r_{HT} \\ e_{\theta\theta} &= \frac{1}{E_\theta}(\sigma_{\theta\theta} - \nu_\theta \sigma_{rr}) + e^\theta_{HT} \\ \gamma_{r\theta} &= \frac{1}{G_{r\theta}}\sigma_{r\theta} \end{cases} \tag{I.57}$$

For the (degenerate) case of isotropic symmetry, $E_r = E_\theta = E, \nu_r = \nu_\theta = \nu, \alpha_r = \alpha_\theta = \alpha$, and $\beta_r = \beta_\theta = \beta$ in which case $e^r_{HT} = e^\theta_{HT} = e^*_{HT}$, as given by (I.43), and the constitutive relations (I.57) redue to those in (I.45) as $G_{r\theta} = G$ for isotropy. The strains $e_{rr}, e_{\theta\theta}$, and $e_{r\theta} = \frac{1}{2}\gamma_{r\theta}$ are still given by the relations following (I.44) or, in terms of the middle surface strains, by

$$\begin{cases} e_{rr} &= e^o_{rr} - \zeta w_{,rr} \\ e_{\theta\theta} &= e^o_{\theta\theta} - \zeta(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta}) \\ \gamma_{r\theta} &= \gamma^o_{r\theta} - 2\zeta(\frac{1}{r}w_{,r\theta} - \frac{1}{r^2}w_{,\theta}) \end{cases} \tag{I.58}$$

where

$$\begin{cases} e^o_{rr} &= u_{r,r} + \frac{1}{2}(w_{,r})^2 \\ e^o_{\theta\theta} &= \frac{1}{r}u_r + \frac{1}{r}u_{\theta,\theta} + \frac{1}{2r^2}(w_{,\theta})^2 \\ \gamma^o_{r\theta} &= u_{\theta,r} - \frac{1}{r}u_\theta + \frac{1}{r}u_{r,\theta} + \frac{1}{r}w_{,r}w_{,\theta} \end{cases} \tag{I.59}$$

Inverting the relations (I.57) we obtain

$$\sigma_{rr} = \frac{E_r}{1-\nu_r\nu_\theta}e_{rr} + \frac{\nu_r E_\theta}{1-\nu_r\nu_\theta}e_{\theta\theta} - \frac{1}{1-\nu_r\nu_\theta}\{E_r e^r_{HT} + \nu_r E_\theta e^\theta_{HT}\} \tag{I.60a}$$

$$\sigma_{\theta\theta} = \frac{\nu_\theta E_r}{1-\nu_r\nu_\theta}e_{rr} + \frac{E_\theta}{1-\nu_r\nu_\theta}e_{\theta\theta} - \frac{1}{1-\nu_r\nu_\theta}\{\nu_\theta E_r e^r_{HT} + E_\theta e^\theta_{HT}\} \tag{I.60b}$$

$$\sigma_{r\theta} = G_{r\theta}\gamma_{r\theta} \tag{I.60c}$$

For the cylindrically orthotropic case, the averaged stresses and bending moments are still defined by the relations following (I.46); thus, by (I.60a, b, c)

$$\begin{cases} N_r &= \frac{E_r h}{1-\nu_r\nu_\theta}e^o_{rr} + \frac{\nu_r E_\theta h}{1-\nu_r\nu_\theta}e^o_{\theta\theta} - N^r_{HT} \\ N_\theta &= \frac{\nu_\theta E_r h}{1-\nu_r\nu_\theta}e^o_{rr} + \frac{E_\theta h}{1-\nu_r\nu_\theta}e^o_{\theta\theta} - N^\theta_{HT} \\ N_{r\theta} &= 2G_{r\theta}he^o_{r\theta} \end{cases} \tag{I.61}$$

where

$$\begin{cases} N^r_{HT} &= \frac{E_r}{1-\nu_r\nu_\theta}\int_{-h/2}^{h/2} e^r_{HT}dz + \frac{\nu_r E_\theta}{1-\nu_r\nu_\theta}\int_{-h/2}^{h/2} e^\theta_{HT}dz \\ N^\theta_{HT} &= \frac{\nu_\theta E_r}{1-\nu_r\nu_\theta}\int_{-h/2}^{h/2} e^r_{HT}dz + \frac{E_\theta}{1-\nu_r\nu_\theta}\int_{-h/2}^{h/2} e^\theta_{HT}dz \end{cases} \tag{I.62}$$

Also, with D_r and D_θ, respectively, the bending stiffnesses around axes in the r and θ directions, passing through a given point in the plate, and $\tilde{D}_{r\theta}$ the twisting rigidity,

$$\begin{cases} D_r &= E_r h^3/12(1-\nu_r\nu_\theta) \\ D_\theta &= E_\theta h^3/12(1-D_r\nu_\theta) \\ \tilde{D}_{r\theta} &= G_{r\theta}h^3/12 \end{cases} \tag{I.63}$$

and

$$D_{r\theta} = D_r \nu_\theta + 2\tilde{D}_{r\theta} \tag{I.64}$$

We also have

$$\begin{cases} M_r &= -D_r[w_{,rr} + \nu_\theta(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta}) - M_{HT}^r \\ M_\theta &= -D_\theta[\nu_r w_{,rr} + (\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta})] - M_{HT}^\theta \\ M_{r\theta} &= -2\tilde{D}_{r\theta}(\frac{w}{r})_{,r\theta} \end{cases} \tag{I.65}$$

with

$$\begin{cases} M_{HT}^r &= \frac{E_r}{1-\nu_r\nu_\theta}\int_{-h/2}^{h/2} e_{HT}^r z dz + \frac{\nu_r E_\theta}{1-\nu_r\nu_\theta}\int_{-h/2}^{h/2} e_{HT}^\theta z dz \\ M_{HT}^\theta &= \frac{\nu_\theta E_r}{1-\nu_r\nu_\theta}\int_{-h/2}^{h/2} e_{HT}^r z dz + \frac{E_\theta}{1-\nu_r\nu_\theta}\int_{-h/2}^{h/2} e_{HT}^\theta z dz \end{cases} \tag{I.66}$$

The expressions in (I.61), for the averaged stresses, and (I.65), for the bending moments, generalize, for the case of cylindrical (polar) orthotropic behavior, the corresponding relations (II.93) and (II.94) in [1] for the non-hygrothermal case.

The first of the generalized von Karman equations for a plate possessing cylindrically orthotropic symmetry is obtained by substituting the expressions in (I.65), (I.66), for the bending moments, into the polar coordinate equivalent form of (I.18), namely,

$$\begin{aligned} &\frac{1}{r}(rM_r)_{,rr} + \frac{1}{r^2}M_{\theta,\theta\theta} - \frac{1}{r}M_{\theta,r} \\ &+\frac{1}{r}M_{r\theta,r\theta} + N_r w_{,rr} \\ &+N_\theta(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta}) + 2N_{r\theta}(\frac{w}{r})_{,r\theta} = 0 \end{aligned} \tag{I.67}$$

where the stress resultants N_r, N_θ, and $N_{r\theta}$ are, again, given by (I.49) in terms of the Airy stress function $\Phi(r,\theta)$. Equation (I.67) is identical with (II.106) of [1]. Carrying out the process indicated, above, we obtain

$$
\begin{aligned}
& D_r w_{,rrrr} + 2D_{r\theta}\frac{1}{r^2}w_{,rr\theta\theta} + D_\theta\frac{1}{r^4}w_{,\theta\theta\theta\theta} \\
& +2D_r\frac{1}{r}w_{,rrr} - 2D_{r\theta}\frac{1}{r^3}w_{,r\theta\theta} - D_\theta\frac{1}{r^2}w_{,rr} \\
& +2(D_\theta + D_{r\theta})\frac{1}{r^4}w_{,\theta\theta} + D_\theta\frac{1}{r^3}w_{,r} \\
& = (\frac{1}{r}\Phi_{,r} + \frac{1}{r^2}\Phi_{,\theta\theta})w_{,rr} \\
& +\Phi_{,rr}(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta}) \\
& +2(\frac{1}{r^2}\Phi_{,\theta} - \frac{1}{r}\Phi_{,r\theta})(\frac{1}{r}w_{,r\theta} - \frac{1}{r^2}w_{,\theta}) \\
& +\frac{1}{r}(rM^r_{HT})_{,rr} + \frac{1}{r^2}(M^\theta_{HT})_{,\theta\theta} - \frac{1}{r}(M^\theta_{HT})_{,r}
\end{aligned}
\tag{I.68}
$$

In order to obtain the polar coordinate equivalent form of the compatibility relation (I.40), without transforming this relation directly, we proceed as follows: with respect to the linearized strains

$$
\begin{cases}
e^\ell_{rr} = u_{r,r} \\
e^\ell_{\theta\theta} = \frac{1}{r}u_r + \frac{1}{r}u_{\theta,\theta} \\
\gamma^\ell_{r\theta} = u_{\theta,r} + \frac{1}{r}u_{r,\theta} - \frac{1}{r}u_\theta
\end{cases}
\tag{I.69}
$$

it is easy to check that

$$
(r\gamma^\ell_{r\theta,\theta})_{,r} - e^\ell_{rr,\theta\theta} - (r^2 e^\ell_{\theta\theta,r})_{,r} + re^\ell_{rr,r} = 0 \tag{I.70a}
$$

Moreover, in view of (I.59),

$$
\begin{cases}
e^o_{rr} = e^\ell_{rr} + \frac{1}{2}(w_{,r})^2 \\
e^o_{\theta\theta} = e^\ell_{\theta\theta} + \frac{1}{2r^2}(w_{,\theta})^2 \\
\gamma^o_{r\theta} = \gamma^\ell_{r\theta} + \frac{1}{r}w_{,r}w_{,\theta}
\end{cases}
\tag{I.70b}
$$

Thus, strain compatibility, i.e. (I.70a), when written in terms of the middle surface strains, requires that

$$
\begin{aligned}
&(r\gamma^o_{r\theta,\theta})_{,r} - e^o_{rr,\theta\theta} - (r^2 e^o_{\theta\theta,r})_{,r} + r e^o_{rr,r} \\
&\quad = [r(\frac{1}{r}w_{,r}w_{,\theta})_{,\theta}]_{,r} - [\frac{1}{2}(w_{,r})^2]_{,r} \\
&\quad -[r^2(\frac{1}{2r^2}(w_{,\theta})^2_{,r}]_{,r} + r(\frac{1}{2}(w_{,r})^2)_{,r}
\end{aligned}
\tag{I.71}
$$

Expanding the right-hand side of (I.71), and simplifying, we obtain

$$
\begin{aligned}
&(r\gamma^o_{r\theta,\theta})_{,r} - e^o_{rr,\theta\theta} - (r^2 e^o_{\theta\theta,r})_{,r} + r e^o_{rr,r} \\
&= w_{,rr}(rw_{,r} + w_{,\theta\theta}) - (w_{,r\theta} - \frac{1}{r}w_{,\theta})^2
\end{aligned}
\tag{I.72}
$$

If we invert the relations in (I.47) we find that

$$
\left\{
\begin{aligned}
e^o_{rr} &= \frac{1}{E_r h}\{(N_r + N^r_{HT}) - \nu_r(N_\theta + N^\theta_{HT})\} \\
e^o_{\theta\theta} &= \frac{1}{E_{\theta h}}\{(N_\theta + N^\theta_{HT}) - \nu_\theta(N_r + N^r_{HT})\} \\
\gamma^o_{r\theta} &= \frac{1}{G_{r\theta}h}N_{r\theta}
\end{aligned}
\right.
\tag{I.73}
$$

Computing, in succession, therefore, the expressions on the left-hand side of the compatibility relation (I.72), we now obtain, through the use of (I.73)

$$
(r\gamma^o_{r\theta,\theta})_{,r} = \frac{1}{G_{r\theta}h}\{N_{r\theta,\theta} + rN_{r\theta,r\theta}\} \tag{I.74a}
$$

$$
e^o_{rr,\theta\theta} = \frac{1}{E_r h}\{(N_{r,\theta\theta} + N^r_{HT,\theta\theta}) - \nu_r(N_{\theta,\theta\theta} + N^\theta_{HT,\theta\theta})\} \tag{I.74b}
$$

$$
\begin{aligned}
(r^2 e^o_{\theta\theta,r})_{,r} &= \frac{r^2}{E_\theta h}\{N_\theta + N^\theta_{HT})_{,rr} - \nu_\theta(N_r + N^r_{HT})_{,rr}\} \\
&+ \frac{2r}{E_\theta h}\{(N_\theta + N^\theta_{HT})_{,r} - \nu_\theta(n_r + N^r_{HT})_{,r}\}
\end{aligned}
\tag{I.74c}
$$

and

$$re^o_{rr,r} = \frac{r}{E_r h}\{(N_r + N^r_{HT})_{,r} - \nu_r(N_\theta + N^\theta_{HT})_{,r}\} \tag{I.74d}$$

Finally, by substituting (I.74a, b, c, d) into (I.72), and simplifying, we obtain

$$\begin{aligned}
&\frac{1}{E_r h}[(N_r - \nu_r N_\theta)_{,\theta\theta} + r(N_{r,r} - \nu_r N_{\theta,r})] \\
&\qquad + \frac{1}{2G_{r\theta}h}(N_{r\theta,\theta} - rN_{r\theta,r\theta}) \\
&+ \frac{1}{E_\theta h}[(r^2(N_{\theta,rr} - \nu_\theta N_{r,rr}) + 2r(N_{\theta,r} - \nu_\theta N_{r,r})] \\
&= w_{,rr}(rw_{,r} + w_{,\theta\theta}) - (w_{,r\theta} - \frac{1}{r}w_{,\theta})^2 \\
&- \frac{1}{E_r h}[(N^r_{HT} - \nu_r N^\theta_{HT})_{,\theta\theta} + r(N^r_{HT,,r} - \nu_r N^\theta_{HT,r})] \\
&\qquad - \frac{1}{E_\theta h}[r^2(N^\theta_{HT,rr} - \nu_\theta N^r_{HT,rr}) \\
&\qquad\qquad + 2r(N^\theta_{HT,r\theta} - \nu_\theta N^r_{HT,r})]
\end{aligned} \tag{I.75}$$

To rewrite (I.75) in terms of the Airy function $\Phi(r,\theta)$, we substitute for N_r, N_θ, and $N_{r\theta}$ in (I.75) from (I.49), multiply the resulting equation through by $-h/r^2$, and simplify; there results:

$$
\begin{aligned}
&\frac{1}{E_\theta}\Phi_{,rrrr} + \left(\frac{1}{G_{r\theta}} - \frac{2\nu_r}{E_r}\right)\frac{1}{r^2}\Phi_{,rr\theta\theta} \\
&+\frac{1}{E_r}\frac{1}{r^4}\Phi_{,\theta\theta\theta\theta} + \frac{2}{E_\theta}\frac{1}{r}\Phi_{,rrr} \\
&-\left(\frac{1}{G_{r\theta}} - \frac{2\nu_r}{E_r}\right)\frac{1}{r^3}\Phi_{,r\theta\theta} - \frac{1}{E_r}\frac{1}{r^2}\Phi_{,rr} \\
&+\left(2\frac{1-\nu_r}{E_r} + \frac{1}{G_{r\theta}}\right)\frac{1}{r^4}\Phi_{,\theta\theta} + \frac{1}{E_r}\frac{1}{r^3}\Phi_{,r} \\
&= -h[w_{,rr}\left(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta}\right) \\
&\quad -\left(\frac{1}{r}w_{,r\theta} - \frac{1}{r^2}w_{,\theta}\right)^2] \\
&+\frac{1}{E_r}\left[\frac{1}{r^2}(N^r_{HT} - \nu_r N^\theta_{HT})_{,\theta\theta} + \frac{1}{r}(N^r_{HT,r} - \nu_r N^\theta_{HT,r})\right] \\
&+\frac{1}{E_\theta}\left[(N^\theta_{HT,rr} - \nu_\theta N^r_{HT,rr}) + \frac{2}{r}(N^\theta_{HT,r\theta} - \nu_\theta N^\theta_{HT,r})\right]
\end{aligned}
\tag{I.76}
$$

Therefore, the generalized von Karman equations governing the bending and buckling of (hygrothermal) cylindrically orthotropic, linearly elastic plates consist of (I.68) and (I.76); in the absence of hygrothermal strains these equations reduce to (II.98), (II.99) of [1]. In (II.68), (I.75) M^r_{HT}, M^θ_{HT} are given by (I.66) and N^r_{HT}, N^θ_{HT} by (I.62) where e^r_{HT}, e^θ_{HT} are defined by (I.56). For a radially symmetric problem all derivatives with respect to θ in both (I.68) and (I.76) would be deleted.

In [1] we commented on the difficulty involved in writing down the von Karman equations which govern the load buckling of a circular (or annular) rectilinearly orthotropic elastic plate; we noted that the structure of the von Karman equations is greatly complicated by the inherent mismatch between the type of elastic symmetry which is built into the form of the constitutive relations and the geometry of the region undergoing buckling. Before we close out this section, we will outline the procedure that must be followed in order to deduce the generalized von Karman equations for bending and buckling of circular (or annular) rectilinearly orthotropic, thin, elastic plates.

The starting points for a derivation of the generalized von Karman equations associated with a rectilinearly orthotropic elastic plate in polar coordinates are the equilibrium equation (I.67) and the compatibility equation (I.72). In (I.67), the averaged stresses $N_r, N_\theta, N_{r\theta}$, may still be expressed, a priori, in terms of the Airy function $\Phi(r,\theta)$ by (I.49). The moments must be expressed in terms of polar coordinates; this means that given $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ (as defined by (I.29))) appropriate expressions for $\sigma_{rr}, \sigma_{\theta\theta}$, and $\sigma_{r\theta}$ must be computed through the use of the transformations preceding (I.45). In order to effectively use the transformation from $(\sigma_{xx}, \sigma_{yy}, \sigma_{xy})$ to $(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta})$, the former set of stress components must be expressed entirely in terms of polar coordinates. As has been noted in [1], the components of the middle surface strain tensors in polar and rectilinear coordinates are related by

$$\left\{\begin{array}{rcl} e^o_{rr} & = & \cos^2\theta e^o_{xx} + \sin^2\theta \cdot e^o_{yy} - \sin\theta\cos\theta\gamma^o_{xy} \\ e^o_{\theta\theta} & = & \sin^2\theta e^o_{xx} + \cos^2\theta \cdot e^o_{yy} - \sin\theta\cos\theta\gamma^o_{xy} \\ \gamma^o_{r\theta} & = & \sin 2\theta(e^o_{yy} - e^o_{xx}) + \cos 2\theta \cdot \gamma^o_{xy} \end{array}\right. \tag{I.77}$$

Inverting the relations in (I.77) we readily obtain

$$\begin{aligned} \mathcal{D}(\theta)\epsilon^o_{xx} = (\cos^2\theta\cos 2\theta + \frac{1}{2}\sin^2 2\theta)e^o_{rr} \\ -(\sin^2\theta\cos 2\theta + \frac{1}{2}\sin^2 2\theta)e^o_{\theta\theta} \\ +\frac{1}{2}\sin 2\theta\cos 2\theta\gamma^o_{r\theta} \end{aligned} \tag{I.78a}$$

$$\begin{aligned} \mathcal{D}(\theta)\epsilon^o_{yy} = (\sin^2\theta\cos 2\theta - \frac{1}{2}\sin^2 2\theta)e^o_{rr} \\ -(\cos^2\theta\cos 2\theta - \frac{1}{2}\sin^2 2\theta)e^o_{\theta\theta} \\ -\frac{1}{2}\sin 2\theta\cos 2\theta \cdot \gamma^o_{r\theta} \end{aligned} \tag{I.78b}$$

and

$$\mathcal{D}(\theta)\gamma^o_{xy} = \sin 2\theta(e^o_{rr} - e^o_{\theta\theta}) + \cos 2\theta \cdot \gamma^o_{r\theta} \tag{I.78c}$$

where $\mathcal{D}(\theta) = \cos^2 2\theta$; we also note that

$$w_{,x} = w_{,r}\cos\theta - \frac{1}{r}w_{,\theta}\sin\theta \tag{I.79}$$

$$w_{,y} = w_{,r}\sin\theta + \frac{1}{r}w_{,\theta}\cos\theta \tag{I.80}$$

so that

$$\begin{aligned} w_{,xx} = \cos^2\theta \cdot w_{,rr} + \sin^2\theta(\frac{1}{r^2}w_{,\theta\theta} + \frac{1}{r}w_{,r}) \\ + \sin 2\theta(\frac{1}{r^2}w_{,\theta} - \frac{1}{r}w_{,r\theta}) \end{aligned} \tag{I.81a}$$

$$\begin{aligned} w_{,yy} = \sin^2\theta \cdot w_{,rr} + \cos^2\theta(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta}) \\ + \sin 2\theta(\frac{1}{r}w_{,r\theta} - \frac{1}{r^2}w_{,\theta}) \end{aligned} \tag{I.81b}$$

and

$$\begin{aligned} w_{,xy} = \sin^2\theta(\frac{1}{r^2}w_{,\theta} - \frac{1}{r}w_{\theta r}) \\ + \cos^2\theta(\frac{1}{r}w_{,r\theta} - \frac{1}{r^2}w_{,\theta}) \\ + \sin\theta\cos\theta(w_{,rr} - \frac{1}{r}w_{,r} - \frac{1}{r^2}w_{,\theta\theta}) \end{aligned} \tag{I.81c}$$

To proceed, we note, e.g., that for fixed but arbitrary θ, σ_{rr} is a linear function of σ_{xx}, σ_{yy}, and σ_{xy}, i.e.,

$$\sigma_{rr} = \cos^2\theta \cdot \sigma_{xx} + \sin^2\theta \cdot \sigma_{yy} + \sin 2\theta \cdot \sigma_{xy} \tag{I.82}$$

while, by (I.29) and (I.7), we may write

$$\begin{aligned} \sigma_{xx} = c_{11}\epsilon^o_{xx} - \zeta c_{11}w_{,xx} - c_{11}\epsilon^1_{HT} \\ + c_{12}\epsilon^o_{yy} - \zeta c_{12}w_{,yy} - c_{12}\epsilon^2_{HT} \end{aligned} \tag{I.83a}$$

$$\begin{aligned} \sigma_{yy} = c_{21}\epsilon^o_{xx} - \zeta c_{21}w_{,xx} - c_{21}\epsilon^1_{HT} \\ + c_{22}\epsilon^o_{yy} - \zeta c_{22}w_{,yy} - c_{22}\epsilon^2_{HT} \end{aligned} \tag{I.83b}$$

and

$$\sigma_{xy} = c_{66}\gamma^o_{xy} - 2\zeta c_{66} w_{,xy} \tag{I.83c}$$

where the c_{ij} are defined by the relations (I.27).

The expressions for $\epsilon^o_{xx}, \epsilon^o_{yy}$, and γ^o_{xy} in (I.78a, b, c) and for $w_{,xx}, w_{,xy}$ and $w_{,xy}$ in (I.18a, b, c)are now substituted into (I.83a, b, c) in order to compute σ_{xx}, σ_{yy}, and σ_{xy} in polar coordinates, after which these expressions are substituted into (I.82) and the analogous relations with respect to $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$. The (resultant) forms of $\sigma_{rr}, \sigma_{\theta\theta}$, and $\sigma_{r\theta}$ are then employed so as to compute M_r, M_θ, and $M_{r\theta}$ which, subsequently, are used in the equilibrium equation (I.67) in order to produce the first of the generalized von Karman equations for hygrothermal bending and buckling. Complete details will be presented in a future report.

In order to derive the second generalized von Karman equation, in polar coordinates, for circular or annular plates exhibiting rectilinear orthotropic response, one uses the compatibility relation (I.72) as a starting point. The stress components $\sigma_{rr}, \sigma_{\theta\theta}$, and $\sigma_{r\theta}$ are expressed in terms of the middle surface strains $e^o_{rr}, e^o_{\theta\theta}$, and $c^o_{r\theta}$, and the hygrothermal strains ϵ^1_{HT} and ϵ^2_{HT}, and the resulting equations are then integrated over the thickness of the plate, and inverted, so as to yield equations analogous to (I.73) for the cylindrically orthotropic case; the relevant second generalized von Karman equation results from the substitution of the forms of $e^o_{rr}, e^o_{\theta\theta}$, and $e^o_{r\theta}$ in (I.72), the introduction of the Airy stress function $\Phi(r, \theta)$, and subsequent simplification. Alternatively, the relevant forms of the generalized von Karman equations for this case may be obtained by directly transforming (I.38) and (I.42) into their polar coordinate equivalent forms.

II. BOUNDARY CONDITIONS

The discussion of boundary conditions for thin elastic plates undergong hygrothermal expansion or contraction parallels that for edge loaded plates in [1]; the important difference is that the expressions for the (resultant) forces and moments along the edge of a plate strains. As in [1], if the thin plate occupies a region Ω in the x, y plane with a smooth (or piecewise smooth) boundary $\partial\Omega$, we let $\vec{n}$ denote the unit normal to the boundary, at any arbitrary point on the boundary, while $\vec{t}$ denotes the unit tangent vector to the boundary at that point. The normal derivative of a function f on $\partial\Omega$ is denoted by $\dfrac{\partial f}{\partial n} \equiv f_{,n}$ while the tangential derivative is given by $\dfrac{\partial f}{\partial s}$, s being a measure of arc length along the boundary. Thus, e.g., if Ω is a disk centered at $(0, 0)$ of radius $R > 0$ and $f = f(r, \theta)$ is defined on Ω and is of class $C^1(\Omega)$, with first derivatives continuous up to $\partial\Omega$, then $\dfrac{\partial f}{\partial n} = f_{,r}$ while $\dfrac{\partial f}{\partial s} = \dfrac{1}{r} f_{,\theta}$. As has been previously noted in [1], the three most prevalent types of boundary conditions in the buckling literature are those which correspond to clamped edges, simply supported edges, and free edges; regardless of whether we are considering plates with isotropic or orthotropic symmetry (either cylindrical or rectilinear), the basic forms assumed by these various sets of boundary conditions are still the same as those delineated in (II.113a, b, c) of [1], i.e.,

(i) $\partial\Omega$ is clamped: $w = 0$ and $\dfrac{\partial w}{\partial n} = 0$, on $\partial\Omega$

(ii) $\partial\Omega$ is simply supported: $w = 0$ and $M_n = 0$, on $\partial\Omega$

(iii) $\partial\Omega$ is free: $M_n = 0$ and $Q_n + \dfrac{\partial M_{tn}}{\partial s} = 0$, on $\partial\Omega$

where M_n is the bending moment on $\partial\Omega$ in the direction normal to $\partial\Omega$, M_{tn} is the twisting moment on $\partial\Omega$, with respect to the tangential and normal directions on $\partial\Omega$, and Q_n is the shearing force associated with the direction normal to $\partial\Omega$.

For the work to be considered in this report only rectangular and circular (or annular) domains Ω will be considered. If $\partial\Omega$ is clamped, therefore, or if, as in the case of a rectangular plate, one or more edges are clamped, the pertinent boundary conditions will be exactly the same as for the non-hygrothermal case, i.e., for a rectangle of width a and length b, exhibiting isotropic response, the clamped boundary conditions are expressed by (II.114a, b) of [1]. For a circular plate (or annular plate) exhibiting isotropic response the relevant conditions are those in (II.123) of [1]. Conditions (II.114a, b) of [1] apply equally well to a rectangular plate exhibiting rectilinear orthotropic response when all the edges are clamped while (II.123) of

[1] still applies for circular (or annular) plates with clamped edge(s) when the plate exhibits cylindrically orthotropic behavior.

We now consider thin elastic plates, subject to hygrothermal expansion or contraction, which have one or more edges simply supported. For a rectangular plate of width a and length b the condition $M_n = 0$ on $\partial\Omega$ becomes

$$\begin{cases} M_x = 0, \quad x = 0, x = a; \quad 0 \le y \le b \\ M_y = 0, \quad y = 0, y = b; \quad 0 \le x \le a \end{cases} \tag{II.1}$$

Thus, by virtue of (I.12),

$$\begin{gathered} K(w_{,xx} + \nu w_{,yy}) + M_{HT} = 0, \\ x = 0, x = a; 0 \le y \le b \end{gathered} \tag{II.2a}$$

and

$$\begin{gathered} K(w_{,yy} + \nu w_{,xx}) + M_{HT} = 0 \\ y = 0, y = b; 0 \le x \le a, \end{gathered} \tag{II.2b}$$

where

$$\begin{cases} M_{HT} &= \dfrac{E}{1-\nu}\displaystyle\int_{-h/2}^{h/2} \epsilon_{HT} z dz \\ \epsilon_{HT} &= \beta\delta H(x,y,z) + \alpha\delta T(x,y,z) \end{cases} \tag{II.3}$$

For the same rectangular elastic plate, now assumed to exhibit rectilinear orthotropic symmetry, the conditions in (II.1) take, as a direct consequence of (I.37) and (I.39), the following form:

$$\begin{gathered} (D_{11} w_{,xx} + D_{12} w_{,yy}) + \tilde{M}^1_{HT} = 0, \\ x = 0, x = a; 0 \le y \le b \end{gathered} \tag{II.4a}$$

$$\begin{gathered} (D_{21} w_{,xx} + D_{22} w_{,yy}) + \tilde{M}^2_{HT} = 0, \\ y = 0, y = b; 0 \le x \le a \end{gathered} \tag{II.4b}$$

where $D_{ij} = c_{ij}h^3/12$, the constitutive constants c_{ij} are given by (I.27) and

$$\tilde{M}^1_{HT} = c_{11} \int_{-h/2}^{h/2} \epsilon^1_{HT}(x,y,z) z dz$$
$$+ c_{12} \int_{-h/2}^{h/2} \epsilon^2_{HT}(x,y,z) z dz \tag{II.5a}$$

$$\tilde{M}^2_{HT} = c_{21} \int_{-h/2}^{h/2} \epsilon^1_{HT}(x,y,z) z dz$$
$$+ c_{22} \int_{-h/2}^{h/2} \epsilon^2_{HT}(x,y,z) z dz \tag{I.5b}$$

with, as per (I.26),

$$\begin{cases} \epsilon^1_{HT} &= \beta_1 \delta H(x,y,z) + \alpha_1 \delta T(x,y,z) \\ \epsilon^2_{HT} &= \beta_2 \delta H(x,y,z) + \alpha_2 \delta T(x,y,z) \end{cases}$$

For an annular elastic plate with (circular) boundaries at $r = R_i, i = 1,2, R_1 = a, R_2 = b > a$, exhibiting isotropic material symmetry, the condition $M_n = 0$ on $\partial\Omega$ translates into

$$M_r = K[w_{,rr} + \nu(\frac{1}{r^2}w_{,\theta\theta} + \frac{1}{r}w_{,r})]$$
$$+ M^*_{HT} = 0, r = R_i, i = 1,2. \tag{II.6}$$

where

$$\begin{cases} M^*_{HT} &= \dfrac{E}{1-\nu} \displaystyle\int_{-h/2}^{h/2} e^*_{HT} z dz \\ e^*_{HT} &= \beta \delta H(r,\theta,z) + \alpha \delta T(r,\theta,z) \end{cases} \tag{II.7}$$

On the other hand, for a circular (or annular) plate with edge(s) at $r = R_i, i = 1,2$, which exhibits cylindrically orthotropic symmetry, we have as the expression of $M_n = 0$ on $\partial\Omega$ the condition

$$D_r[w_{,rr} + \nu_\theta(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta})] + M^r_{HT} = 0,$$
$$r = R_i, i = 1,2 \tag{II.8}$$

where, by virtue of (I.66) and (I.56),

$$\begin{cases} M^r_{HT} = \dfrac{E_r}{1-\nu_r\nu_\theta}\displaystyle\int_{-h/2}^{h/2} e^r_{HT} dz + \dfrac{\nu_r E_\theta}{1-\nu_r\nu_\theta}\displaystyle\int_{-h/2}^{h/2} e^\theta_{HT} z dz \\ \begin{cases} e^r_{HT} = \beta_r \delta H(r,\theta,z) + \alpha_r \delta T(r,\theta,z) \\ e^\theta_{HT} = \beta_\theta \delta H(r,\theta,z) + \alpha_\theta \delta T(r,\theta,z) \end{cases} \end{cases}$$

and $D_r = E_r h^3/12(1-\nu_r\nu_\theta)$.

Along any edge of a thin plate which is free, we must have $M_n = 0$ as well as $Q_n + \dfrac{\partial M_{tn}}{\partial s} = 0$. Conditions equivalent to $M_n = 0$ along a portion of $\partial\Omega$ (or all of $\partial\Omega$) for various cases of interest have been elucidated above. For rectangular plates of width a and length b, it has been shown in [1] that the condition $Q_n + \dfrac{\partial M_{tn}}{\partial s} = 0$ on $\partial\Omega$ is equivalent to the following relations

$$\begin{aligned} &M_{y,y} + 2M_{xy,x} = 0, \\ &y = 0, y = b; 0 \le x \le a \end{aligned} \tag{II.9a}$$

$$\begin{aligned} &M_{x,x} + 2M_{yx,y} = 0, \\ &x = 0, x = a; 0 \le y \le b \end{aligned} \tag{II.9b}$$

If the rectangular plate exhibits isotropic response then the bending moments M_x, M_y, and M_{xy} are given by (I.12) in which case (II.9a,b) become

$$\begin{aligned} K[w_{,yyy} + (2-\nu)w_{,xxy}] + M_{HT,y} = 0 \\ y = 0, y = b; 0 \le x \le a \end{aligned} \tag{II.10a}$$

and

$$\begin{aligned} K[w_{,xxx} + (2-\nu)w_{,xyy}] + M_{HT,x} = 0 \\ x = 0, x = a; 0 \le y \le b \end{aligned} \tag{II.10b}$$

with $M_{HT} = \dfrac{E}{1-\nu}\displaystyle\int_{-h/2}^{h/2} \epsilon_{HT} z dz$ and $\epsilon_{HT} = \beta\delta H + \alpha\delta T$.

For the case in which the rectangular plate exhibits rectilinear orthotropic symmetry, (II.9a,b) still represent the conditions equivalent to $Q_n + \dfrac{\partial M_{\epsilon n}}{\partial s} = 0$ along all four edges but, now, the bending moments M_x, M_y, and M_{xy} are given by (I.37). An easy computation then shows that in lieu of (II.10a, b) for the isotropic case we have

$$D_{21}w_{,xxy} + D_{22}w_{,yyy} + 4D_{66}w_{,xxy} + \tilde{M}^2_{HT,y} = 0, \quad y = 0, b; 0 \leq x \leq a \tag{II.11a}$$

and

$$D_{11}w_{,xxx} + D_{12}w_{,xyy} + 4D_{66}w_{,xyy} + \tilde{M}^1_{HT,x} = 0, \quad x = 0, a; 0 \leq y \leq b \tag{II.11b}$$

where $\tilde{M}^1_{HT}, \tilde{M}^2_{HT}$ are defined by (I.36) and (I.39) with $\epsilon^1_{HT}, \epsilon^2_{HT}$ as given by (I.26).

In order to elucidate the free edge boundary conditions which apply with respect to thin elastic plates with a circular geometry, we note that (see [3], §2.2) in the orthogonal (s, n) coordinate system introduced along the boundary $\partial\Omega$ of a domain Ω in the x, y plane

$$Q_n = \frac{\partial M_n}{\partial n} + \frac{\partial M_{tn}}{\partial s} \tag{II.12}$$

Thus, along a free edge we must have $M_n = 0$ as well as

$$\frac{\partial M_n}{\partial n} + 2\frac{\partial M_{tn}}{\partial s} = 0 \tag{II.13}$$

Along the edge at $r = R$ of a circular plate, the condition (II.13) assumes the form

$$M_{r,r} + \frac{2}{r}M_{r\theta,\theta} = 0 \tag{II.14}$$

For an annular plate with edges at $r = R_i, i = 1, 2 (R_1 = a, R_2 = b > a)$, which exhibits isotropic material symmetry, the condition corresponding to (II.14) becomes

$$K\{(w_{,rr} + \frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta})_{,r} + (1-\nu)(\frac{1}{r^2}w_{,\theta\theta r} - \frac{1}{r^3}w_{,\theta\theta})\} + M^*_{HT,r} = 0, r = R_i, i = 1, 2 \tag{II.15}$$

where $M^*_{HT}(r,\theta)$ is given by (II.7). For the same annular plate, this time exhibiting cylindrically orthotropic behavior, we have as a consequence of (I.65)

$$D_r[w_{,rrr} + \nu_\theta(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta})_{,r}]$$
$$+\frac{4}{r}\tilde{D}_{r\theta}(\frac{w}{r})_{,r\theta\theta} + M^r_{HT,r} = 0, \tag{II.16}$$
$$r = R_i, i = 1,2$$

where $D_r = E_r h^3/12(1-\nu_r\nu_\theta)$, $\tilde{D}_{r\theta} = G_{r\theta}h^3/12$, and M^r_{HT} is given by (I.56) and (I.66). Of course, for both the isotropic and cylindrically orthotropic cases we must have $M_r = 0$ along $r = R_i, i = 1,2$, with M_r given by (II.6) in the isotropic case and by (I.65) in the cylindrically orthotropic case. If $R_1 = a = 0$ the annular case reduces to the case of a disk (circular plate of radius $R_2 = b$).

III. THE VON KARMAN EQUATIONS FOR THERMAL BENDING AND BUCKLING OF PLATES

Although the primary motivating factor behind the preparation of this report was (and remains) to prepare a foundation and background material for a study of the hygroexpansive buckling (and curl) of paper, it is a fact that almost all of the literature on hygrothermal bending and buckling of plates has focused on purely thermal problems; in order to survey some of that literature, therefore, we shall now specialize some of the equations and boundary conditions specified in $\S II$ for the mixed hygrothermal situation to the thermal case only. We shall also look at the reductions that occur in the thermal buckling and bending equations when special forms of the temperature distribution in the plate are considered or when we restrict ourselves to the small deflection case or ignore middle surface forces (acting in the plane of the plate); at this point we shall also append to the first of the relevant von Karman equations, in each case, a distributed force $t = t(x,y)$ normal to the middle surface of the plate. In all the cases to be considered in this section an equivalent hygroexpansive (or hygrocontractive) problem results by replacing the thermal expansion coefficients by hygroscopic coefficients and the plate temperature distribution by an equivalent distribution of moisture in the plate.

We begin by specializing the equations and boundary conditions derived in rectilinear coordinates so as to cover the specific case of thermal bending and buckling. Thus, in (I.2), $\beta \equiv 0$ so that

$$\epsilon_{HT} \equiv \epsilon_T = \alpha \delta T(x, y, z) \tag{III.1}$$

where it is assumed that the thermal expansion coefficient α is constant. For isotropic response the constitutive relations (I.9) then reduce to

$$\begin{cases} N_x &= \dfrac{Eh}{1-\nu^2}(\epsilon_{xx}^o + \nu\epsilon_{yy}^o) - N^T \\ N_y &= \dfrac{Eh}{1-\nu^2}(\epsilon_{yy}^o + \nu\epsilon_{xx}^o) - N^T \\ N_{xy} &= 2Gh\epsilon_{xy}^o \end{cases} \tag{III.2}$$

where $\epsilon_{xx}^o, \epsilon_{xy}^o, \epsilon_{yy}^o$ are the middle surface strains as defined by (I.7) and N^T is given by (I.11a). The bending moments are given as

$$\begin{cases} M_x &= -K(w_{,xx} + \nu w_{,yy}) - M^T \\ M_y &= -K(w_{,yy} + \nu w_{,xx}) - M^T \\ M_{xy} &= -(1-\nu)Kw_{,xy} \end{cases} \tag{III.3}$$

with the thermal moment M^T given by (I.14).

The generalized von Karman equations for the hygrothermal case now reduce to (see (I.21), (I.24)):

$$K\triangle^2 w = \Phi_{,yy}w_{,xx} - 2\Phi_{,xy}w_{,xy} + \Phi_{,xx}w_{,yy} - \triangle M^T + t \tag{III.4a}$$

and

$$\triangle^2\Phi = -\frac{1}{2}Eh[w,w] - (1-\nu)\triangle N^T \tag{III.4b}$$

where $t = t(x,y)$ is the applied transverse force. In considering small deflection theory one ignores the bracket operator on the right-hand side of (III.4b), in which case the Airy function, as given by (I.20), satisfies

$$\triangle^2\Phi = -(1-\nu)\triangle N^T(\text{small deflection theory}) \qquad (\overline{III.4b})$$

In a purely (thermal) bending problem the middle surface forces as given by (I.20) do not come into play in which case the system (III.4a, b) reduces to

$$K\triangle^2 w = -\triangle M^T + t \text{ (thermal bending).} \qquad (\overline{III.4a})$$

In many places in the literature a temperature distribution of the form

$$\delta T(x,y,z) = T_o(x,y) + zT_1(x,y) \qquad \text{(III.5)}$$

has been considered; for this specific type of distribution it is easy to see that, as a direct consequence of (I.11a) and (I.14),

$$\left\{\begin{array}{lcl} N^T & = & \dfrac{\alpha E H}{1-\nu}T_o(x,y) \\ M^T & = & \dfrac{\alpha E}{1-\nu}\cdot\dfrac{h^3}{12}T_1(x,y) \end{array}\right. \qquad \text{(III.6)}$$

For such a temperature distribution, within the context of the small deflection equations, it will generally be the case that one is dealing with a thermal bending problem when $\triangle N^T = \triangle T_o = 0$ and a thermal buckling problem when $\triangle M^T = \triangle T_1 = 0$. The boundary conditions associated with the thermal bending and buckling of, say, a rectangular plate of width a and length b exhibiting isotropic material symmetry are as follows:

(i) If all the edges are clamped then the conditions coincide with (II.114a, b) of [1].

(ii) If all the edges are simply supported then $w = 0$ along each edge and, in addition,

$$\begin{cases} K(w_{,xx} + \nu w_{,yy}) + M^T = 0 \\ \qquad x = 0, a; 0 \leq y \leq b \\ K(w_{,yy} + \nu w_{,xx}) + M^T = 0 \\ \qquad y = 0, b; 0 \leq x \leq a \end{cases} \tag{III.7}$$

(iii) If the edges are all free then the conditions in (III.7) hold as well as

$$\begin{cases} K[w_{,yyy} + (2 - \nu) w_{,xxy}] + M^T_{,y} = 0 \\ \qquad y = 0, b; 0 \leq x \leq a \\ K[w_{,xxx} + (2 - \nu) w_{,xyy}] + M^T_{,x} = 0 \\ \qquad x = 0, a; 0 \leq y \leq b \end{cases} \tag{III.8}$$

In (III.7), (III.8), M^T is given by (I.14). Of course, cases where, e.g., one pair of (parallel) edges is simply supported while the other pair of edges is free can be considered by combining the conditions in (i)-(iii), above.

For a plate exhibiting rectilinearly orthotropic material symmetry the thermal strains assume the form (see (I.26)):

$$\epsilon_T^1 = \alpha_1 \delta T(x, y, z), \epsilon_T^2 = \alpha_2 \delta T(x, y, z) \tag{III.9}$$

with the coefficients of linear thermal expansion along the x and y axes, α_1 and α_2, respectively, taken to be constants. The constitutive relations in this situation are given as follows (where $\epsilon^o_{xx}, \epsilon^o_{xy}, \epsilon^o_{yy}$ are, once again, the middle surface strains):

$$\begin{cases} N_x &= \{\frac{E_1 h}{1 - \nu_{12}\nu_{21}}\}\epsilon^o_{xx} + \}\frac{E_2 \nu_{21} H}{1 - \nu_{12}\nu_{21}}\}\epsilon^o_{yy} - \tilde{N}^1_T \\ N_y &= \{\frac{E_1 \nu_{12} h}{1 - \nu_{12}\nu_{21}}\}\epsilon^o_{xx} + \{\frac{E_2 h}{1 - \nu_{12}\nu_{21}}\}\epsilon^o_{yy} - \tilde{N}^2_T \\ N_{xy} &= (2G_{12}h)\epsilon^o_{xy} \end{cases} \tag{III.10}$$

where (see (I.31)):

$$\begin{cases} \tilde{N}_T^1 &= (\dfrac{E_1\alpha_1 + E_2\alpha_2\nu_{21}}{1-\nu_{12}\nu_{21}}) \displaystyle\int_{-h/2}^{h/2} \delta T(x,y,z)dz \\ \tilde{N}_T^2 &= (\dfrac{E_1\alpha_1\nu_{12} + E_2\alpha_2}{1-\nu_{12}\nu_{21}}) \displaystyle\int_{-h/2}^{h/2} \delta T(x,y,z)dz \end{cases} \tag{III.11}$$

The expressions for the bending moments, in the thermal bending/buckling problem for a rectilinearly orthotropic elastic plate, assume the form (see (I.37)):

$$\begin{cases} M_x &= -(D_{11}w_{,xx} + D_{12}w_{,yy}) - \tilde{M}_T^1 \\ M_y &= -(D_{21}w_{,xx} + D_{22}w_{,yy}) - \tilde{M}_T^2 \\ M_{xy} &= -2D_{66}w_{,xy} \end{cases} \tag{III.12}$$

where

$$\begin{cases} \tilde{M}_T^1 &= (c_{11}\alpha_1 + c_{12}\alpha_2) \displaystyle\int_{-h/2}^{h/2} \delta T(x,y,z)zdz \\ \tilde{M}_T^2 &= (c_{21}\alpha_1 + c_{22}\alpha_2) \displaystyle\int_{-h/2}^{h/2} \delta T(x,y,z)zdz \end{cases} \tag{III.13}$$

and the c_{ij} are given by (I.27). The relevant von Karman equations for thermal bending/buckling become (see (I.38), (I.42)):

$$\begin{aligned} &D_{11}w_{,xxxx} + \{D_{12} + 4D_{66} + D_{21}\}w_{,xxyy} \\ &+D_{22}w_{,yyyy} = [\Phi, w] - \tilde{M}^1_{T,xx} - \tilde{M}^1_{T,yy} + t \end{aligned} \tag{III.14a}$$

and

$$\begin{aligned} \frac{1}{E_1 h}\Phi_{,yyyy} &+ \frac{1}{h}(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_2})\Phi_{,xxyy} \\ &+\frac{1}{E_2 h}\Phi_{,xxxx} = -\frac{1}{2}[w,w] \\ &-\frac{1}{E_1 h}(\tilde{N}_T^1 - \nu_{21}\tilde{N}_T^2)_{,yy} \\ &-\frac{1}{E_2 h}(\tilde{N}_T^2 - \nu_{12}\tilde{N}_T^1)_{,xx} \end{aligned} \tag{III.14b}$$

As for the boundary data in this case, with respect, e.g., to a rectangular plate of width a and length b, we have the following:

(i) If all four edges are clamped then conditions (III.114a, b) of [1] still apply.

(ii) If the four edges are simply supported, then $w = 0$ along each edge and, in addition,

$$\begin{cases} (D_{11}w_{,xx} + D_{12}w_{,yy}) + \tilde{M}_T^1 = 0, \\ \qquad\qquad x = 0, a; 0 \leq y \leq b \\ (D_{21}w_{,xx} + D_{22}w_{,yy}) + \tilde{M}_T^2 = 0, \\ \qquad\qquad y = 0, b; 0 \leq x \leq a \end{cases} \tag{III.15}$$

where $\tilde{M}_T^1, \tilde{M}_T^2$ are given by (III.13).

(iii) If all four edges are free, then the conditions in (III.15) apply as well as

$$\begin{cases} D_{21}w_{,xxy} + D_{22}w_{,yyy} + 4D_{66}w_{,xxy} + \tilde{M}_{T,y}^2 = 0, \\ \qquad\qquad y = 0, b; 0 \leq x \leq a \\ D_{11}w_{,xxx} + D_{12}w_{,xyy} + 4D_{66}w_{,xyy} = \tilde{M}_{T,x}^1 = 0 \\ \qquad\qquad x = 0, a; 0 \leq y \leq b \end{cases} \tag{III.16}$$

The same comments apply as in the isotropic case with regard to different types of boundary data holding along pairs of parallel edges of the plate; in a small deflection situation, the 'bracket' term $-\frac{1}{2}[w, w]$ would be deleted from the right-hand side of (III.14b) while for purely thermal buckling equation (III.14b) would be deleted in its entirety and (III.14a) would be employed with $\Phi \equiv 0$.

When the temperature difference δT varies linearly through the thickness of the plate, as in (III.5), the thermal stress resultants $\tilde{N}_T^i, i = 1, 2$ and the thermal moments $\tilde{M}_T^i, i = 1, 2$, as given by (III.11) and (III.13), respectively, reduce to

$$\begin{cases} \tilde{N}_T^1 &= (\dfrac{E_1\alpha_1 + E_2\alpha_2\nu_{21}}{1-\nu_{12}\nu_{21}})hT_o(x,y) \\ \tilde{N}_T^2 &= (\dfrac{E_1\alpha_1\nu_{12} + E_2\alpha_2}{1-\nu_{12}\nu_{21}})hT_o(x,y) \end{cases} \tag{III.17a}$$

and

$$\left\{ \tilde{M}_T^1 = \frac{(c_{21}\alpha_1 + c_{22}\alpha_2)h^3}{12} T_1(x,y) \right. \tag{III.17b}$$

For a circular (or annular) plate exhibiting isotropic material symmetry, the thermal strain (see (I.43))) is given by

$$e_T^* = \alpha\delta T(r,\theta,z) \tag{III.18}$$

For the isotropic case, the constitutive relations (I.47) reduce to

$$\begin{cases} N_r &= \dfrac{Eh}{1-\nu^2}(e_{rr}^o + \nu e_{\theta\theta}^o) - N_T^* \\ N_\theta &= \dfrac{Eh}{1-\nu^2}(e_{\theta\theta}^o + \nu e_{rr}^o) - N_T^* \\ N_{r\theta} &= 2Ghe_{r\theta}^o \end{cases} \tag{III.19}$$

where the $e_{rr}^o, e_{r\theta}^o, e_{\theta\theta}^o$ are the middle surface strains as given by (II.71) of [1] with $\zeta = 0$, while

$$N_T^* = \frac{E\alpha}{1-\nu}\int_{-h/2}^{h/2} \delta T(r,\theta,z)dz \tag{III.20}$$

The bending moments for the isotropic case in polar coordinates are given by (see [3], §4):

$$\begin{cases} M_r &= -K[w_{,rr} + \nu(\dfrac{1}{r}w_{,r} + \dfrac{1}{r^2}w_{,\theta\theta})] - M_T^* \\ M_\theta &= -K[\nu w_{,rr} + (\dfrac{1}{r}w_{,r} + \dfrac{1}{r^2}w_{,\theta\theta})] - M_T^* \\ M_{r\theta} &= -(1-\nu)K(w_{,r\theta} - \dfrac{1}{r^2}w_{,\theta}) \end{cases} \tag{III.21}$$

where

$$M_T^* = \frac{E\alpha}{1-\nu}\int_{-h/2}^{h/2} \delta T(r,\theta,z)zdz \tag{III.22}$$

The von Karman system for this case now assumes the form

$$\begin{cases} K\triangle^2 w &= [\Phi, w] - \triangle M_T^* + t(r,\theta) \\ \triangle^2\Phi &= -\frac{1}{2}Eh[w,w] - (1-\nu)\triangle N_T^* \end{cases} \tag{III.23}$$

where $[\Phi, w]$ and $[w, w]$, as well as $\triangle^2 w$, are given by the expressions directly following (I.51) and $\triangle = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$; for the small deflection case one again deletes the 'bracket' $[w, w]$ in the second equation in (III.23). Thermal bending alone for the circular (or annular) isotropic plate is governed by the first of the two equations in (III.23) with $\Phi \equiv 0$.

For a temperature distribution varying linearly through the thickness of the plate we have, in lieu of (III.5),

$$\delta T(r,\theta,z) = T_o(r,\theta) + zT_1(r,\theta) \tag{III.24}$$

In this special case (III.20) and (III.22) become, respectively,

$$\begin{cases} N_T^* &= \frac{E\alpha h}{1-\nu} T_o(r,\theta) \\ M_T^* &= \frac{E\alpha h^3}{12(1-\nu)} T_1(r,\theta) \end{cases} \tag{III.25}$$

We now delineate the boundary conditions that are associated with the system (III.23), or a specialization thereof, for the case of an annular plate with edges at $r = R_i, i = 1, 2, R_1 = a, R_2 = b > a$.

(i) If both edges are clamped then $w = 0$ and $\frac{\partial w}{\partial r} = 0$, along $r = R_i, i = 1, 2$.

(ii) If the plate edges at $r = R_i, i = 1, 2$ are simply supported then $w = 0$, for $r = R_i, i = 1, 2$, and, in addition,

$$K[w_{,rr} + \nu(\frac{1}{r^2}w_{,\theta\theta} + \frac{1}{r}w_{,r})] + M_T^* = 0 \quad \text{for } r = R_i, i = 1, 2 \tag{III.26}$$

(iii) When the edges at $r = R_i, i = 1, 2$ are free we must use (III.26) as well as the condition

$$K\left\{(w_{,rr} + \frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta})_{,r} + (1-\nu)(\frac{1}{r^2}w_{,\theta\theta r} - \frac{1}{r^3}w_{,\theta\theta})\right\} + M^*_{T,r} = 0, r = R_i, i = 1, 2 \tag{III.27}$$

Our last case in this sequence concerns the thermal bending and/or buckling of thin elastic circular (or annular) plates exhibiting cylindrically orthotropic behavior. The thermal strains in the radial and angular directions are given by (see (I.56))

$$e^r_T = \alpha_r \delta T(r, \theta, Z), e^\theta_T = \alpha_\theta \delta T(r, \theta, z) \tag{III.28}$$

In lieu of (III.19), (III.20) and (III.21), (III.22) for the isotropic case we now have the following sets of expressions for the resultant forces and bending moments:

$$\begin{cases} N_r = \frac{E_r h}{1-\nu_r\nu_\theta}e^o_{rr} + \frac{\nu_r E_\theta h}{1-\nu_r\nu_\theta}e^o_{\theta\theta} - N^r_T \\ N_\theta = \frac{\nu_\theta E_r h}{1-\nu_r\nu_\theta}e^o_{rr} + \frac{E_\theta h}{1-\nu_r\nu_\theta}e^o_{\theta\theta}N^\theta_T \\ N_{r\theta} = 2G_{r\theta}he^o_{r\theta} \end{cases} \tag{III.29}$$

with

$$\begin{cases} N^r_T = (\frac{E_r\alpha_r + \nu_r E_\theta\alpha_\theta}{1-\nu_r\nu_\theta})\int_{-h/2}^{h/2}\delta T(r,\theta,z)dz \\ N^\theta_T = (\frac{\nu_\theta E_r\alpha_r + E_\theta\alpha_\theta}{1-\nu_r\nu_\theta})\int_{-h/2}^{h/2}\delta T(r,\theta,z)dz \end{cases} \tag{III.30}$$

and

$$\begin{cases} M_r = -D_r[w_{,rr} + \nu_\theta(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta})] - M^r_T \\ M_\theta = -D_\theta[\nu_r w_{,rr} + (\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta})] - M^\theta_T \\ M_{r\theta} = -2\tilde{D}_{r\theta}(\frac{w}{r})_{,r\theta} \end{cases} \tag{III.31}$$

with

$$\left\{ \begin{array}{lcl} M_T^r & = & (\frac{E_r\alpha_r + \nu_r E_\theta \alpha_\theta}{1-\nu_r\nu_\theta}) \int_{-h/2}^{h/2} \delta T(r,\theta,z) z dz \\ M_T^\theta & = & (\frac{\nu_\theta E_r \alpha_r + E_\theta \alpha_\theta}{1-\nu_r\nu_\theta}) \int_{-h/2}^{h/2} \delta T(r,\theta,z) z dz \end{array} \right. \tag{III.32}$$

Employing (III.31) in (I.67), and using (I.49), we obtain the first of the von Karman equations for thermal bending and/or buckling of a thin, elastic, cylindrically orthotropic plate, i.e.,

$$\begin{aligned} & D_r w_{,rrrr} + 2D_{r\theta} \cdot \frac{1}{r^2} w_{,rr\theta\theta} + D_\theta \cdot \frac{1}{r^4} w_{,\theta\theta\theta\theta} \\ & +2D_r \cdot \frac{1}{r} w_{,rrr} - 2D_{r\theta} \frac{1}{r^3} w_{,r\theta\theta} - D_\theta \frac{1}{r^2} w_{,rr} \\ & +2(D_\theta + D_{r\theta}) \frac{1}{r^4} w_{,\theta\theta} + D_\theta \frac{1}{r^3} w_{,r} \\ & = (\frac{1}{r}\Phi_{,r} + \frac{1}{r^2}\Phi_{,\theta\theta}) w_{,rr} \\ & +\Phi_{,rr}(\frac{1}{r} w_{,r} + \frac{1}{r^2} w_{,\theta\theta}) \\ & +2(\frac{1}{r^2}\Phi_{,\theta} - \frac{1}{r}\Phi_{,r\theta})(\frac{1}{r} w_{,r\theta} - \frac{1}{r^2} w_{,\theta}) \\ & +\frac{1}{r}(rM_T^r)_{,rr} + \frac{1}{r^2}(M_T^\theta)_{,\theta\theta} \\ & -\frac{1}{r}(M_T^\theta)_{,r} + t(r,\theta) \end{aligned} \tag{III.33}$$

while the second of the relevant von Karman equations for this case becomes

$$
\begin{aligned}
&\frac{1}{E_\theta}\Phi_{,rrrr}+(\frac{1}{G_{r\theta}}-\frac{2\nu_r}{Er})\frac{1}{r^2}\Phi_{,rr\theta\theta}\\
&+\frac{1}{E_r}\cdot\frac{1}{r^4}\Phi_{,\theta\theta\theta\theta}+\frac{2}{E_\theta}\cdot\frac{1}{r}\Phi_{,rrr}\\
&-(\frac{1}{G_{r\theta}}-\frac{2\nu_r}{E_r})\frac{1}{r^3}\Phi_{,r\theta\theta}-\frac{1}{E_r}\cdot\frac{1}{r^2}\Phi_{,rr}\\
&+(2\frac{1-\nu_r}{E_r}+\frac{1}{G_{r\theta}})\frac{1}{r^4}\Phi_{,\theta\theta}+\frac{1}{E_r}\cdot\frac{1}{r^3}\Phi_{,r}\\
&=-h[w_{,rr}(\frac{1}{r}w_{,r}+\frac{1}{r^2}w_{,\theta\theta})-(\frac{1}{r}w_{,r\theta}-\frac{1}{r^2}w_{,\theta})^2]\\
&+\frac{1}{E_r}\{\frac{1}{r^2}(N_T^r-\nu_r N_T^\theta)_{,\theta\theta}+\frac{1}{r}(N_{T,r}^r-\nu_r N_{T,r}^\theta)\}\\
&+\frac{1}{E_\theta}\{(N_{T,r}^\theta-\nu_\theta N_{T,rr}^r)+\frac{2}{r}(N_{T,r\theta}^\theta-\nu_\theta N_{T,r}^\theta)\}
\end{aligned}
\tag{III.34}
$$

with N_T^r and N_T^θ as given by (III.30).

Remarks: Various combinations of terms on the right-hand side of (III.34) may be simplified somewhat, e.g.,

$$N_T^r-\nu_r N_T^\theta=E_r\alpha_r\int_{-h/2}^{h/2}\delta T(r,\theta,z)dz$$

but there appears to be little value in carrying out such an exercise except within the context of an application to a specific problem.

Among the special cases of the von Karman system (III.33), (III.34) that are of particular interest are the following:

(i) when the temperature distribution varies linearly through the plate, as in (III.24), the thermal moments and stress resultants in (III.33) and (III.34) reduce to the following expressions:

$$
\left\{
\begin{aligned}
N_T^r &= (\frac{E_r\alpha_r+\nu_r E_\theta\alpha_\theta}{1-\nu_r\nu_\theta})hT_o(r,\theta)\\
N_T^\theta &= (\frac{\nu_\theta E_r\alpha_r+E_\theta\alpha_\theta}{1-\nu_r\nu_\theta})hT_o(r,\theta)
\end{aligned}
\right.
\tag{III.35}
$$

and

$$\begin{cases} M_T^r &= (\dfrac{E_r\alpha_r + \nu_r E_\theta \alpha_\theta}{1-\nu_r\nu_\theta})\dfrac{h^3}{12}\cdot T_1(r,\theta) \\ M_T^\theta &= (\dfrac{\nu_\theta E_r\alpha_r + E_\theta \alpha_\theta}{1-\nu_r\nu_\theta})\dfrac{h^3}{12}\cdot T_1(r,\theta) \end{cases} \tag{III.36}$$

(ii) For small deflections the first term on the right-hand side of (III.34), in square brackets, is deleted.

(iii) The (purely) thermal bending problem is governed by (III.33) with $\Phi = 0$.

(iv) For an axially symmetric problem both (III.33) and (III.34) reduce to (variable coefficient) ordinary differential equations in the radial variable r; we have $w = w(r), \Phi = \Phi(r)$ and $T = T(r,z)$ so that the resultants N_T^r, N_T^θ and moments M_T^r, M_T^θ are functions only of r. In (III.33) and (III.34), therefore, derivatives of all quantities, of any order, with respect to θ vanish.

The boundary data for an annular elastic plate of inner radius $R_1 = a$ and outer radius $R_2 = b > a$, exhibiting cylindrically orthotropic symmetry, may be specified as follows:

(i) If both edges are clamped then $w = 0$ and $\dfrac{\partial w}{\partial r} = 0$, along $r = R_i, i = 1,2$.

(ii) If the plate edges are simply supported, then $w = 0$, for $r = R_i, i = 1,2$ and, additionally,

$$D_r[w_{,rr} + \nu_\theta(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta})] + M_T^r = 0 \qquad \text{for } r = R_i, i = 1,2 \tag{III.37}$$

where M_T^r is given by the first of the relations in (III.32).

(iii) For free edges at $r = R_i, i = 1,2$, (III.37) applies and, in addition,

$$\begin{aligned} &D_r[w_{,rrr} + \nu_\theta(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta})_{,r}] \\ &+\frac{4}{r}\tilde{D}_{r\theta}(\frac{w}{r})_{,r\theta} + M^r_{T,r} = 0, \\ &\text{for } r = R_i, i = 1,2 \end{aligned} \tag{III.38}$$

The usual considerations apply if one edge is, e.g., clamped while the other is simply supported or if one edge is simply supported while the other is free, etc.; for a circular plate of radius $R = b, R_1 = a = 0$.

Remarks: All of the problems considered above may be posed in terms of the middle surface displacements u, v and the out-of-plane deflection w in lieu of w and the Airy function Φ; the idea is most feasible within the context of small deflection theory. For an isotropic rectangular plate small deflection theory corresponds to the substitution of the expressions for the middle surface strains $\epsilon^o_{xx}, \epsilon^o_{xy}$, and ϵ^o_{yy} from (I.7) into the constitutive relations (III.2), suppression of all those terms involving the out-of-plane displacement w, and then substitution of the resultant expressions for N_x, N_y, and N_{xy} into the in-plane equilibrium equations (I.17a); this process leads to the pair of equations

$$\left\{ \begin{array}{l} \dfrac{Eh}{1-\nu^2}(u_{,xx} + \nu v_{,xy}) + \dfrac{Eh}{2(1+\nu)}(u_{,yy} + v_{,xy}) = N^T_{,x} \\ \\ \dfrac{Eh}{2(1+\nu)}(u_{,xy} + v_{,xx}) + \dfrac{Eh}{1-\nu^2}(\nu u_{,xy} + v_{,yy}) = N^T_{,y} \end{array} \right. \tag{III.39}$$

which must be solved in conjunction with

$$K\triangle^2 w = N_x w_{,xx} + N_{xy} w_{,xy} + N_y w_{,yy} - \triangle M^T + t \tag{III.40}$$

For the small deflection case considered, above, (III.39) and (III.40) are decoupled, just as (III.4a), (III.4b) are if, in (III.4b), we delete the bracket $[w, w]$. Thus (III.39) must be solved, subject to appropriate boundary conditions, for the middle surface displacements u, v which, in turn, are used to compute N_x, N_{xy}, and N_y for subsequent substitution in (III.40). Little use of the displacement formulation of the thermal bending/buckling problem will be made in this report and we will, therefore, not pursue the issue further with respect to other geometries or other classes of material symmetry.

Remarks: All of the thermal bending/buckling equations, and corresponding boundary conditions considered in this section, may be derived from energy principles, i.e., from the principle of minimum potential energy in conjunction with elementary techniques in the calculus of variations; we will illustrate the general idea for an isotropic plate in rectangular Cartesian coordinates. Energy principles also serve as the basis for various approximate methods of analysis, including the Rayleigh-Ritz method (for computing critical (buckling) temperatures and the corresponding (initial) buckling modes) and finite element methods. In what follows we will consider only the thermal bending problem for the sake of simplifying, somewhat, the presentation.

The standard descriptions of the principle of minimum potential energy within the context of structural mechanics is as follows: of all displacement fields which satisfy the prescribed constraint conditions, the state assumed by the structure is the one which makes the total potential energy a minimum.

For an elastic plate, the total potential energy Π is the sum of the strain energy U and the potential of any (conservative) applied forces. For the case of an isotropic, linearly elastic plate the strain energy U, within the context of small deflection (classical plate) theory assumes the form

$$U = \int_{\mathcal{A}}\int \{ \frac{Eh}{2(1-\nu^2)}(u_{,x}+v_{,y})^2 + \frac{Eh}{4(1+\nu)}[(u_{,y}+v_{,x})^2$$

$$-4u_{,x}v_{,y}] + \frac{K}{2}(w_{,xx}+w_{,yy})^2$$

$$+(1-\nu)K[w_{,xy}^2 - w_{,xx}w_{,yy}]$$

$$-N^T(u_{,x}+v_{,y}) + M^T(w_{,xx}+w_{,yy})\} dxdy,$$

where $\mathcal{A}$ is the area of the middle surface of the plate, while the potential of the transverse loading $t(x,y)$ is

$$V = -\int_{\mathcal{A}}\int twdxdy$$

Therefore for a rectangular plate $(0 \leq x \leq a, 0 \leq y \leq b)$ which is subject to a temperature variation $\delta T(x,y,z)$ and a transverse loading $t(x,y)$, but no applied edge loads, the total potential energy Π assumes the form:

$$\Pi = \int_o^b \int_o^a \{\frac{Eh}{2(1-\nu^2)}(u_{,x}+v_{,y})^2 + \frac{Eh}{4(1+\nu)}[(u_{,y}+v_{,x})^2$$
$$-4u_{,x}v_{,y}] + \frac{K}{2}(w_{,xx}+w_{,yy})^2$$
$$+(1-\nu)K[w_{,xy}^2 - w_{,xx}w_{,yy}] \qquad \text{(III.41)}$$
$$-N^T(u_{,x}+v_{,y}) + M^T(w_{,xx}+w_{,yy}) - tw\}dxdy$$

To apply the principle of minimum potential energy we note that a necessary condition for Π to have a minimum is that the first variation $\delta\Pi = 0$. Using (III.41) to compute $\delta\Pi$, integrating by parts, and employing (III.2) and (III.3), with $\epsilon^o_{xx}, \epsilon^o_{xy}, \epsilon^o_{yy}$ as given by classical plate theory, we find that

$$
\begin{aligned}
\delta\Pi = \int_o^b \int_o^a \{ &-[\frac{Eh}{1-\nu^2}(u_{,xx} + \nu v_{,xy}) \\
&+\frac{Eh}{2(1+\nu)}(u_{,yy} + v_{,xy}) - N^T_{,x}]\delta u \\
&-[\frac{Eh}{2(1+\nu)}(u_{,xy} + v_{,xx}) + \frac{Eh}{1-\nu^2}(\nu u_{,xy} + v_{,yy}) - N^t_{,y}]\delta v \\
&+[K(w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}) \\
&+M^T_{,xx} + M^T_{,yy} - t]\delta w\} dxdy \\
&+\int_o^b \{[N_x \delta u]_o^a + [N_{xy}\delta v]_o^a \\
&+[(M_{x,x} + 2M_{xy,y})\delta w]_o^a - [M_x \delta(w_{,x})]_o^a\} dy \\
&+\int_o^a \{[N_x \delta u]_o^b + [N_y \delta v]_o^b + [(M_{y,y} + 2M_{y,x})\delta w]_o^b \\
&-[M_y \delta(w_{,y})]_o^b\} dx \\
&-[2M_{xy}\delta w]_{(o,b)}^{(a,b)} + [2M_{xy}\delta w]_{(o,o)}^{(a,o)}
\end{aligned}
\tag{III.42}
$$

In order that (III.42) be satisfied for arbitrary variations $\delta u, \delta v, \delta w, \delta(w_{,x})$, and $\delta(w_{,y})$, all the expressions within the square brackets in (III.42) must vanish identically which leads to (III.39) and (III.40), with $N_x = N_{xy} = N_y = 0$; the resultant system governs the thermal bending, under an arbitrary temperature distribution $\delta T(x, y, z)$, of a linearly elastic, isotropic rectangular plate. From (III.42) it is readily deduced that the following (natural) boundary conditions must be satisfied:

(i) On the edges $x = 0, a$, for $0 \leq y \leq b$

$$\begin{cases} u \text{ is prescribed or } N_x = 0 \\ v \text{ is prescribed or } N_{xy} = 0 \\ w \text{ is prescribed or } M_{x,x} + 2M_{xy,y} = 0 \\ w_{,x} \text{ is prescribed or } M_x = 0 \end{cases} \tag{III.43a}$$

(ii) On the edges $y = 0, b$, for $0 \leq x \leq a$

$$\begin{gathered} u \text{ is prescribed or } N_{xy} = 0 \\ v \text{ is prescribed or } N_y = 0 \\ w \text{ is prescribed or } M_{y,y} + 2M_{xy,x} = 0 \end{gathered} \tag{III.43b}$$

(iii) At the corners $(0, 0), (a, 0), (0, b)$, and (a, b)

$$w \text{ is prescribed or } M_{xy} = 0 \tag{III.43c}$$

Remarks: Some consideration of the calculation of thermal stress distributions will be made in § IV, while problems of buckling, bending, and postbuckling for rectangular plates and plates with circular symmetry will be treated at length in §V; however, it is feasible to present here the simple, but important problem of a thin plate (of arbitrary contour) which is subjected to a temperature distribution that varies only through the thickness of the plate, i.e., $T = T(z)$. In this case we clearly have that N^T and M^T are constants. We will restrict the discussion to the case of an isotropic, linearly elastic, plate (in rectangular Cartesian coordinates) which is either free or has clamped edges.

For the case of a free plate, a solution to (III.4b), with $[w, w] = 0$, which yields zero force resultants on the boundary is given by $\Phi = 0$. It then follows that $N_x = N_y = N_{xy} = 0$

throughout the plate. Inversion of (III.2) yields

$$\begin{cases} \epsilon^o_{xx} &= \dfrac{1}{Eh}[N_x - \nu N_y + (1-\nu)N^T] \\ \epsilon^o_{yy} &= \dfrac{1}{Eh}[N_y - \nu N_x + (1-\nu)N^T] \\ \epsilon^o_{xy} &= \dfrac{1+\nu}{Eh}N_{xy} \end{cases} \tag{III.44}$$

Integration of (III.44) yields (recall that $\epsilon^o_{xx}, \epsilon^o_{xy}$, and ϵ^o_{yy} are the classical plate theory middle surface strains) the in-plane displacements:

$$\begin{cases} u &= \dfrac{(1-\nu)N^T}{Eh}x + a + cy \\ v &= \dfrac{(1-\nu)N^T}{Eh}y + b - cx \end{cases} \tag{III.45}$$

with a, b, c arbitrary constants of integration. In a similar vein, if we take $M_x = M_y = M_{xy} = 0$, throughout the plate, then the boundary conditions for a free edge are automatically satisfied while (III.4) has as its solution (recall that $\Phi = 0$ and we also take $t \equiv 0$)

$$w = -\frac{M^T}{2(1+\nu)K}(x^2 + y^2) + d + ex + fy \tag{III.46}$$

with d, e, and f constants of integration. The resulting thermal stresses for this case are easily computed to be (e.g., [3], § 2.4)

$$\sigma_{xx} = \sigma_{yy} = \frac{1}{h}N^T + \frac{12}{h^3}M^T z - \frac{E\alpha T(z)}{1-\nu}, \ \sigma_{xy} = 0 \tag{III.47}$$

For the case where the plate has clamped edges, instead of free edges, it again follows that a simple solution exists. With constant M^T, and $t \equiv 0$, (III.4a) and the boundary conditions are satisfied by taking $w = 0$. Then, by virtue of (III.3),

$$M_x = M_y = -M^T, M_{xy} = 0 \tag{III.48}$$

In in-plane edge displacements are prevented then equations (III.39) and the boundary conditions yield $u = v = 0$ so that, as a consequence of (III.2),

$$N_x = N_y = -N^T, N_{xy} = 0 \tag{III.49}$$

In this situation it is easily computed that

$$\sigma_{xx} = \sigma_{yy} = -\frac{E\alpha T(z)}{1-\nu}, \ \sigma_{xy} = 0 \tag{III.50}$$

If, on the other hand, the middle surface of the plate is free of in-plane tractions, then $N_x = N_y = N_{xy} = 0$ and

$$\sigma_{xx} = \sigma_{yy} = \frac{1}{h}N^T - \frac{E\alpha T(z)}{1-\nu}, \ \sigma_{xy} = 0 \tag{III.51}$$

IV. THERMAL BENDING AND BUCKLING OF RECTANGULAR AND CIRCULAR PLATES AND THERMOELASTIC STRESS DISTRIBUTIONS–SMALL DEFLECTION THEORY

There are many excellent surveys of thermoelastic problems in the mechanics literature (e.g., Boley and Weiner [7], Nowacki [8], Hetnarski [9], and Kovalenko [10]) to which the reader may be referred. In this section, we will content ourselves with presenting only a few thermal stress distribution solutions which have been considered in conjunction with problems involving the thermal bending and/or buckling of thin plates within the context of small deflection theory; the associated bending and/or buckling solutions are also presented and analyzed.

Within the context of small-deflection theory, two distinct types of problems may be considered: those in which the effect, on the deflections, of loads in the plane of the plate is neglected, thus leading to a thermal bending problem and those in which the effect of such loads is taken into account thereby leading to a buckling problem; postbuckling behavior can not be adequately accounted for in the context of small-deflection theory and, therefore, descriptions of thermal postbuckling behavior are relegated to §V and § VI.

If the effect of loads in the plane of the plate on deflections is ignored, then for the simplest case of an isotropic rectangular plate subjected to a transverse loading $t = t(x, y)$, and a general three-dimensional temperature variation $\delta T(x, y, z)$, the pertinent equation is (III.4a) with $\Phi \equiv 0$, i.e.,

$$K(w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}) = t - M^t_{,xx} - M^T_{,yy} \tag{IV.1}$$

Equation (IV.1) is to hold for $0 < x < a, 0 < y < b$. For illustrative purposes, we will assume that the plate is simply supported on all four edges, in which case $w = 0$ for $x = 0, a, 0 \leq y \leq b, w = 0$ for $y = 0, b, 0 \leq x \leq a$, and the conditions (III.7) apply as well. The analysis proceeds by expressing the thermal moment and transverse load as double Fourier sine series of the form

$$\begin{cases} M^T & = \displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} M_{mn} \sin \alpha_m \sin \beta_n y \\ t & = \displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} t_{mn} \sin \alpha_m \sin \beta_n y \end{cases} \tag{IV.2}$$

with $\alpha_m = \dfrac{m\pi}{a}, \beta_n = \dfrac{n\pi}{b}$ and

$$(M_{mn}, t_{mn}) = \frac{4}{ab}\int_o^b \int_o^a (M^T, t) \sin \alpha_m \sin \beta_n y dx dy$$

In order to satisfy the boundary conditions we take the deflection to have the form

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \gamma_{mn} \sin \alpha_m \sin \beta_n y \tag{IV.3}$$

Thus, in accord with (III.7) it must be assumed that the temperature distribution is such that $M^T = 0$, for $x = 0, a, 0 \leq y \leq b$ and $M^T = 0$ for $y = 0, b, 0 \leq x \leq a$, a situation that we will comment on at length in sections V and VI.

Substituting (IV.2) and (IV.3) into (IV.1) and solving for the coefficients γ_{mn} we have

$$\gamma_{mn} = \frac{t_{mn} + (\alpha_m^2 + \beta_n^2) M_{mn}}{K(\alpha_m^2 + \beta_n^2)^2} \tag{IV.4}$$

so that the resultant moments, as a consequence of (III.3), (IV.2) - (IV.4), become

$$\begin{cases} M_x &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{K(\alpha_m^2+\nu\beta_n^2)\gamma_{mn} - M_{mn}\}\sin\alpha_m \sin\beta_n y \\ M_y &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{K(\nu\alpha_m^2+\beta_n^2)\gamma_{mn} - M_{mn}\}\sin\alpha_m \sin\beta_n y \\ M_{xy} &= -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(1-\nu)K\alpha_m\beta_n\gamma_{mn}\cos\alpha_m \cos\beta_n y \end{cases} \tag{IV.5}$$

To resolve the in-plane stretching aspect of this problem we note that the displacements u, v are governed by the differential equations (III.39) with which we may associate two types of boundary conditions: in the first case it may be assumed that normal components of displacements along each edge are permitted while tangential components are not while in the second case the opposite situation would prevail. For the first case alluded to, the boundary data takes on the form

$$\begin{cases} N_x &= \dfrac{Eh}{1-\nu^2}(u_{,x}+\nu v_{,y} - N^T = 0, v = 0 \\ & \qquad \text{for } x = 0, a; 0 \leq y \leq 6 \\ N_y &= \dfrac{Eh}{1-\nu^2}(\nu u_{,x}+v_{,y}) - N^T = 0, u = 0 \\ & \qquad \text{for } y = 0, b; 0 \leq x \leq a \end{cases} \tag{IV.6}$$

Of course, small deflection theory has been assumed in writing down (IV.6). We now express the thermal force N^T as

$$N^T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} N_{mn}\sin\alpha_m \sin\beta_n y \tag{IV.7}$$

with

$$N_{mn} = \frac{4}{ab}\int_o^b \int_o^a N^T \sin\alpha_m \sin\beta_n y dx dy,$$

N^T being given by (I.11a) for a general variation $\delta T(x, y, z)$.

To satisfy the boundary conditions (IV.6) we seek solutions of (III.39) in the form

$$\left\{\begin{array}{lcl} u & = & \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\cos\alpha_m \sin\beta_n y \\ v & = & \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mn}\sin\alpha_m \cos\beta_n y \end{array}\right. \tag{IV.8}$$

The expressions in (IV.8) are now substituted into the equations (III.39) with the result that

$$\left\{\begin{array}{lcl} a_{mn} & = & \dfrac{-\alpha_m N_{mn}(1-\nu^2)}{Eh(\alpha_m^2+\beta_n^2)} \\ b_{mn} & = & \dfrac{-\beta_n N_{mn}(1-\nu^2)}{Eh(\alpha_m^2+\beta_n^2)} \end{array}\right. \tag{IV.9}$$

Employing (IV.7) - (IV.8) in (IV.6) we compute, for the stress resultants

$$\left\{\begin{array}{lcl} N_x & = & -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{\dfrac{Eh}{1-\nu^2}(\alpha_m a_{mn}+\nu\beta_n b_{mn}) \\ & & +N_{mn}\}\sin\alpha_m \sin\beta_n y \\ N_y & = & -\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\{\dfrac{Eh}{1-\nu^2}(\nu\alpha_m a_{mn}+\beta_n b_{mn}) \\ & & +N_{mn}\}\sin\alpha_m \sin\beta_n y \\ N_{xy} & = & \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\dfrac{1}{2}\dfrac{Eh}{1+\nu}(\beta_n a_{mn}+\alpha_{nm} b_{mn})\cos\alpha_m\cos\beta_n y \end{array}\right. \tag{IV.10}$$

If, in lieu of the boundary data (IV.6), one assumes that tangential displacements are allowed along each edge, but that normal displacements are prevented, the relevant boundary conditions are

$$N_{xy} = \frac{1}{2}\frac{Eh}{1+\nu}(u_{,y}+v_{,x}) = 0 \tag{IV.11a}$$
$$\text{for } x = 0, a; 0 \leq y \leq b, \text{ and } y = 0, b; 0 \leq x \leq a$$

and

$$u = 0, \text{for } x = 0, a; 0 \leq y \leq b, v = 0, \text{ for } y = 0, b; 0 \leq x \leq a \tag{IV.11b}$$

In this case N^T may be expressed as

$$N^T = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \bar{N}_{mn} \cos\alpha_m \cos\beta_n y \tag{IV.12}$$

with

$$\begin{cases} \bar{N}_{mn} & = & \dfrac{\zeta_{mn}}{ab}\displaystyle\int_o^b\int_o^a N^T \cos\alpha_m \cos\beta_n y dx dy \\ \zeta_{mn} & = & \begin{cases} 4, m > 0, n > 0 \\ 2, m > 0, n = 0 \text{ or } m = 0, n > 0 \\ 1, m = n = 0 \end{cases} \end{cases}$$

while the displacements, chosen so as to identically satisfy the boundary conditions in (IV.11a, b), have the form

$$\begin{cases} u & = & \displaystyle\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \bar{a}_{mn} \sin\alpha_m \cos\beta_n y \\ v & = & \displaystyle\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \bar{b}_{mn} \cos\alpha_m \sin\beta_n y \end{cases} \tag{IV.13}$$

The same procedure described above, for the first set of boundary conditions now leads to

$$\bar{a}_{mn} = \frac{\alpha_m \bar{N}_{mn}(1-\nu^2)}{Eh(\alpha_m^2+\beta_n^2)}, \bar{b}_{mn} = \frac{\beta_n \bar{N}_{mn}(1-\nu^2)}{Eh(\alpha_m^2+\beta_n^2)} \tag{IV.14}$$

and

$$\begin{cases} N_x & = & \displaystyle\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\{\frac{Eh}{1-\nu^2}(\alpha_m\bar{a}_{mn}+\nu\beta_n\bar{b}_{mn}) \\ & & -\bar{N}_{mn}\}\cos\alpha_m\cos\beta_n y \\ N_y & = & \displaystyle\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\{\frac{Eh}{1-\nu^2}(\nu\alpha_m\bar{a}_{mn}+\beta_n\bar{b}_{mn}) \\ & & -\bar{N}_{mn}\}\cos\alpha_m\cos\beta_n y \\ N_{xy} & = & -\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{2}\frac{Eh}{1+\nu}(\beta_n\bar{a}_{mn}+\alpha_m\bar{b}_{mn})\sin\alpha_m\sin\beta_n y \end{cases} \tag{IV.15}$$

The thermoelastic stress distributions, for either of the two bending problems considered above, may be computed by substitution for $N_x, N_{xy}, N_y, N^T, M_x, M_y, M_{xy}$, and M^T in the relations

$$\left\{\begin{array}{lcl} \sigma_{xx} & = & \frac{1}{h}(N_x + N^T) + \frac{12z}{h^3}(M_x + M^T) - \frac{E\alpha}{1-\nu}T \\ \sigma_{yy} & = & \frac{1}{h}(N_y + N^T) + \frac{12z}{h^3}(M_y + M^T) - \frac{E\alpha}{1-\nu}T \\ \sigma_{xy} & = & \frac{1}{h}N_{xy} + \frac{12z}{h^3}M_{xy} \end{array}\right. \tag{IV.16}$$

An alternative solution to the flexure problem for the rectangular plate discussed above has been described in Tauchert [3] and is now described below; we will begin, as in [3], by assuming that the edges $x = 0$ and $x = a$ are simply supported, that the plate is symmetric with respect to the x axis, so that $-\frac{b}{2} \leq y \leq \frac{b}{2}$, and that, for now, the boundary conditions along $y = \pm\frac{1}{2}b$ are arbitrary. As $w = 0$ along $x = 0$ and $x = a$ it follows that $w_{,yy} = 0$ along these edges as well. The conditions of simple support of the plate along $x = 0$ and $x = a$ may, therefore, be expressed as

$$\begin{array}{c} w = 0, w_{,xx} = -\frac{1}{K}M^T \\ \text{for } x = 0, a; 0 \leq y \leq b \end{array} \tag{IV.17}$$

We look for a solution of (IV.1) satisfying the non-homogeneous boundary conditions (IV.17) in the form

$$w = W(x,y) + M^T(0,y)H_o(x) + M^T(a,y)H_a(x) \tag{IV.18}$$

with

$$\left\{\begin{array}{lcl} H_o(x) & = & \frac{a^2}{6K}\{\frac{x}{a} - 3(\frac{x}{a})^2 + (\frac{x}{a})^3\} \\ H_a(x) & = & \frac{a^2}{6K}\{\frac{x}{a} - (\frac{x}{a})^3\} \end{array}\right. \tag{IV.19}$$

Using (IV.19) in (IV.18) and substituting the resultant expression for $w(x,y)$ into (IV.1) it follows that

$$\nabla^4 w(x,y) = F(x,y) \tag{IV.20}$$

where

$$F(x,y) = \frac{t(x,y)}{K} - \frac{\nabla^2 M^T}{K} \\ - \nabla^4 \left\{ M^T(0,y)H_o(x) + M^T(a,y)H_a(x) \right\} \tag{IV.21}$$

and

$$w = \frac{\partial^2 w}{\partial x^2} = 0, \text{ for } x = 0, a; 0 \leq y \leq b \tag{IV.22}$$

We express F in terms of the Fourier series

$$F(x,y) = \sum_{m=1}^{\infty} f_m(y) \sin \alpha_m x; \quad \alpha_m = \frac{m\pi}{\alpha} \tag{IV.23}$$

with

$$f_m(y) = \frac{2}{a} \int_o^a F(x,y) \sin \alpha_m x dx$$

and take w in the form

$$w(x,y) = \sum_{m=1}^{\infty} Y_m(y) \sin \alpha_m x \tag{IV.24}$$

so that w automatically satisfies the edge conditions in (IV.22). It is easily shown that w, as given by (IV.24), satisfies (IV.20) provided the Y_m satisfy the ordinary differential equations

$$Y_m^{(iv)}(y) - 2\alpha_m^2 Y_m''(y) + \alpha_m^4 Y_m(y) = f_m(y) \tag{IV.25}$$

whose general solution has the form

$$Y_m = (A_m + B_m y) \cosh \alpha_m y + (C_m + D_m y) \sinh \alpha_m y \\ + e^{-\alpha_m y} \int e^{2\alpha_m y} (\int e^{-\alpha_m y} f_m(y) dy) dy \tag{IV.26}$$

The constants of integration A_m, B_m, C_m, and D_m in (IV.26) are to be determined from the boundary data on the edges $y = \pm\frac{1}{2}b$. Suppose, e.g., that the edges $y = \pm\frac{1}{2}b$ are also simply supported and that the thermal moment M^T is constant–as it would be, say, for $\delta T = \delta T(z)$. As the support conditions and loading are both symmetric with respect to the x-axis, the deformation must also be symmetric and, thus, $B_m = C_m = 0$ in (IV.26). Substituting Y_m from (IV.26) into (IV.24) and, then, the resultant expression for w into (IV.18) yields

$$\begin{aligned} w = \sum_{m=1}^{\infty} (A_m \cosh \alpha_m y + D_m y \sinh \alpha_m y) \sin \alpha_m x \\ + M^T(o,y) H_o(x) + M^T(a,y) H_a(x) \end{aligned} \tag{IV.27}$$

The sum of the last two terms on the right-hand side of (IV.27) may be expressed as a Fourier series, i.e.,

$$\begin{aligned} & M^T(0,y) H_o(x) + M^T(a,y) H_a(x) \\ & \equiv \frac{M^T a^2}{2K}(\frac{x}{a} - (\frac{x}{a})^2) \equiv \sum_{m=1}^{\infty} k_m \sin \alpha_m x \end{aligned} \tag{IV.28}$$

where

$$k_m = \begin{cases} 0, & m \text{ even} \\ \dfrac{4M^T}{aK\alpha_m^3}, & m \text{ odd} \end{cases}$$

in which case

$$w = \sum_{m=1,3,\cdots}^{\infty} (A_m \cosh \alpha_m y + D_m y \sinh \alpha_m y + k_m) \sin \alpha_m x \tag{IV.29}$$

The constants A_m and D_m in (IV.29) are to be determined from the boundary conditions

$$\begin{aligned} w = 0, w_{,yy} = -\frac{1}{K} M^T \\ \text{for } y = \pm\frac{1}{2}b; 0 \leq x \leq a \end{aligned} \tag{IV.30}$$

We write

$$M^T = \sum_{m=1}^{\infty} M_m \sin \alpha_m x; \; m_m = K\alpha_m^2 k_m \tag{IV.31}$$

and substitute (IV.29) into (IV.30); after solving for A_m and D_m we obtain

$$w(x,y) = \frac{4M^T}{aK} \sum_{m=1,3,\cdots}^{\infty} \frac{1}{\alpha_m^3}(1 - \frac{\cosh \alpha_m y}{\cosh \frac{1}{2}\alpha_m b}) \sin \alpha_m x \tag{IV.32}$$

for which the corresponding moment resultants are given by

$$\left\{\begin{array}{lcl} M_x & = & \dfrac{-4M^T(1-\nu)}{a} \displaystyle\sum_{m=1,3,\cdots}^{\infty} \frac{\cosh \alpha_m y}{\alpha_m \cosh \frac{1}{2}\alpha_m b} \sin \alpha_m x \\ M_y & = & \dfrac{-4M^T(1-\nu)}{a} \displaystyle\sum_{m=1,3,\cdots}^{\infty} \frac{1}{\alpha_m}(1 - \frac{\cosh \alpha_m y}{\cosh \frac{1}{2}\alpha_m b}) \sin \alpha_m x \\ M_{xy} & = & \dfrac{4M^T(1-\nu)}{a} \displaystyle\sum_{m=1,3,\cdots}^{\infty} \frac{1}{\alpha_m}(\frac{\sinh \alpha_m y}{\cosh \frac{1}{2}\alpha_m b}) \cos \alpha_m x \end{array}\right. \tag{IV.33}$$

For the case in which the plate is clamped along the edges at $y = \pm\frac{1}{2}b$, and subject to a constant thermal moment M^T, it has been noted in [3] that the deflection, once again, assumes the form in (IV.29) but, now, with

$$\left\{\begin{array}{lcl} A_m & = & -k_m(\frac{1}{2}\alpha_m b \cosh \frac{1}{2}\alpha_m b \\ & & +\sinh \frac{1}{2}\alpha_m b)/\triangle_m \\ B_m & = & \frac{1}{2}\alpha_m b + \sinh \frac{1}{2}\alpha_m b \cosh \frac{1}{2}\alpha_m b \\ \triangle_m & = & \frac{1}{2}\alpha_m b + \sinh \frac{1}{2}\alpha_m b \cosh \frac{1}{2}\alpha_m b \end{array}\right. \tag{IV.34}$$

References for the thermal bending of an isotropic, elastic rectangular plate, under other combinations of edge conditions, may be found in [3].

Next, we consider the problem of thermal bending of an isotropic annular plate; we assume that the plate is subjected to a transverse loading $t = t(r, \theta)$ and a general temperature variation $\delta T(r, \theta, z)$. For this situation, ignoring for now the effect on deflections of loads in the plane of the plate, the relevant equation is the first partial differential equation in

(III.23) with $\Phi \equiv 0$, i.e.,

$$\begin{aligned} & K(w_{,rrrr} + \frac{2}{r}w_{,rrr} - \frac{1}{r^2}w_{,rr} \\ & + \frac{2}{r^2}w_{,rr\theta\theta} + \frac{1}{r^3}w_{,r} - \frac{2}{r^3}w_{,r\theta\theta} \\ & + \frac{1}{r^4}w_{,\theta\theta\theta\theta} + \frac{4}{r^4}w_{,\theta\theta}) \\ & = t - (M^*_{T,rr} + \frac{1}{r}M^*_{T,r} + \frac{1}{r^2}M^*_{T,\theta\theta}) \end{aligned} \tag{IV.35}$$

with $M^*_T = \dfrac{E\alpha}{1-\nu}\displaystyle\int_{-h/2}^{h/2} \delta T(r,\theta,z) z dz$ being the thermal moment.

Equation (IV.35) holds for $a \leq r < b, 0 \leq \theta < 2\pi$. The associated clamped, simply supported, and free edge boundary conditions are given, respectively, by

(i) $w = 0$ and $\dfrac{\partial w}{\partial r} = 0$, at $r = a, b$ if the edges are clamped

(ii) $w = 0$ and $K[w_{,rr} + \nu(\frac{1}{r^2}w_{,\theta\theta} + \frac{1}{r}w_{,r})] + M^*_T = 0$, at $r = a, b$ if the edges are simply supported

(iii) $K[w_{,rr} + \nu(\frac{1}{r^2}w_{,\theta\theta} + \frac{1}{r}w_{,r})] + M^*_T = 0$ and

$$K[(w_{,rr} + \frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta})_{,r} + (1-\nu)(\frac{1}{r^2}w_{,\theta\theta r} - \frac{1}{r^3}w_{,\theta\theta})] + M^*_{T,r} = 0,$$

at $r = a, b$ if both edges are free.

Also, for isotropic response, the bending moments in polar coordinates are given by (III.21), the resultant forces by (III.19), with $N^*_T = \dfrac{E\alpha}{1-\nu}\displaystyle\int_{-h/2}^{h/2} \delta T(r,\theta,z) dz$ and the stresses may be expressed by

$$\left\{ \begin{aligned} \sigma_{rr} &= \frac{1}{h}(N_r + N^*_T) + \frac{12z}{h^3}(M_r + M^*_T) - \frac{E\alpha}{1-\nu}\delta T \\ \sigma_{\theta\theta} &= \frac{1}{h}(N_\theta + N^*_T) + \frac{12z}{h^3}(M_\theta + M^*_T) - \frac{E\alpha}{1-\nu}\delta T \\ \sigma_{r\theta} &= \frac{1}{h}N_{r\theta} + \frac{12z}{h^3}M_{r\theta} \end{aligned} \right. \tag{IV.36}$$

The simplest case of thermal bending with respect to an annular plate is that of axisymmetric bending in which it is assumed that the loading and boundary conditions are independent of the angular coordinate θ. If, in addition, $t \equiv 0$ then (IV.35) reduces to

$$\nabla^4 w = -\frac{1}{K} \nabla^2 M_T^*, a < r < b \tag{IV.37}$$

where $w = w(r)$, $M_T^* = \dfrac{E\alpha}{1-\nu} \displaystyle\int_{-h/2}^{h/2} \delta T(r,z) z dz$,

$$\begin{cases} \nabla^4 &= \dfrac{d^4}{dr^4} + \dfrac{2}{r}\dfrac{d^3}{dr^3} - \dfrac{1}{r^2}\dfrac{d^2}{dr^2} + \dfrac{1}{r^3}\dfrac{d}{dr} \\ \nabla^2 &= \dfrac{d^2}{dr^2} + \dfrac{1}{r}\dfrac{d}{dr} \equiv \dfrac{1}{r}\dfrac{d}{dr}(r\dfrac{d}{dr}) \end{cases}$$

The general solution of (IV.37) is easily computed to be

$$\begin{aligned} w &= C_1 + C_2 r^2 + C_3 \ln\frac{r}{a} + C_4 r^2 \ln\frac{r}{a} \\ &+ \int_r^b \left(\frac{1}{r} \int_a^r \frac{1}{K} M_T^*(r) r dr\right) dr \end{aligned} \tag{IV.38}$$

with the $C_i, i = 1, \cdots, 4$, arbitrary constants of integration. For the problem at hand a straightforward computation based on (IV.38) yields the following expressions for the relevant moments and shear force resultant:

$$\begin{cases} M_r &= -K\{2(1+\nu)C_2 - (1-\nu)\dfrac{C_3}{r^2} + (3+\nu)C_4 \\ & \quad +2(1+\nu)C_4 \ln\dfrac{r}{a}\} - \dfrac{1-\nu}{r^2} \displaystyle\int_a^r M_T^*(r) r dr \\ M_{r\theta} &= 0 \\ Q_r &\equiv M_{r,r} + \dfrac{M_r - M_\theta}{r} = -4K\dfrac{C_4}{4} \end{cases} \tag{IV.39}$$

For the case of a solid plate, in which $a = 0$, the constants C_3 and C_4 in (IV.38) must vanish so that M_r and Q_r remain finite at $r = 0$; if the solid plate is clamped along its edge at

$r = b$, then it follows from (IV.38), and the fact that $w = w_{,r} = 0$ at $r = b$, that

$$C_1 = -b^2 C_2 = -\frac{1}{2K}\int_0^b M_T^*(r) r dr \tag{IV.40}$$

while if the edge at $r = b$ is simply supported

$$C_1 = -b^2 C_2 = \frac{1-\nu}{2(1+\nu)K}\int_0^b M_T^*(r) r dr \tag{IV.41}$$

When $(\delta T)_{,\theta} \neq 0$ a solution $w = w(r,\theta)$ must be obtained for (IV.35); for simplicity we again set $t \equiv 0$; such problems have, e.g., been considered by Forray and Newman [11] for the special case in which the thermal gradient is assumed to vary linearly through the thickness of the plate. Specifically, it is assumed in [11] that the thermal moment M_T^* may be expressed in the form

$$M_T^* = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} A_{km} r^k \cos m\theta + \sum_{m=1}^{\infty}\sum_{k=0}^{\infty} B_{km} r^k \sin m\theta \tag{IV.42}$$

This form for the thermal moment is a consequence of the assumption that

$$\delta T(r,\theta,z) = T_0(r,\theta) + zT_1(r,\theta)$$

with

$$T_1(r,\theta)z = \frac{-z}{h} T_d(r,\theta) \tag{IV.43}$$

and T_d the temperature difference between the upper and lower faces of the plate. Using the definition of M_T^* we then easily compute that

$$M_T^* = \frac{1+\nu}{hK}\alpha T_d(r,\theta) \tag{IV.44}$$

while (IV.35) becomes (with $t \equiv 0$)

$$\nabla^4 w = \frac{1+\nu}{hK}\alpha \nabla^2 T_d \tag{IV.45}$$

The solution to (IV.45) consists of the sum of the general solution of $\nabla^4 w_g = 0$ and a particular solution of $\nabla^2 w_p = \dfrac{1+\nu}{hK}\alpha T_d$.

In fact, the general solution to (IV.45) can be shown, as in [11], to have the form

$$
\begin{aligned}
w \;=\;& a_0 + b_0 r^2 + c_0 r^2 lnr + d_0 lnr \\
&+(a_1 r + b_1 r^3 + \frac{c_1}{r} + d_1 r lnr)\cos\theta \\
&+(a_1' r + b_1' r^3 + \frac{c_1'}{r} + d_1' r lnr)\sin\theta \\
&+\sum_{n=2}^{\infty}\left(a_n r^n + b_n r^{n+2} + \frac{c_n}{r^n} + \frac{d_n}{r^{n-2}}\right)\cos n\theta \\
&+(a_n' r^n + b_n' r^{n+2} + \frac{c_n'}{r_n} + \frac{d_n'}{r^{n-2}})\sin n\theta \\
&+\sum_{m=0}^{\infty} q_m(r)\cos m\theta + \sum_{m=1}^{\infty} h_m(r)\sin m\theta
\end{aligned}
\tag{IV.46}
$$

with the $a_n, a_n', b_n, \ldots (n = 0, 1, \ldots)$ arbitrary constants and

$$
(g_m, h_m) = -\frac{1}{Kr^m}\int\left(r^{2m-1}\int (A_{km}, b_{km}) r^{k+1-m} dr\right) dr \tag{IV.47}
$$

For the special case in which we are dealing with a solid plate, so that $a = 0$, we must set $c_n = c_n' = d_n = d_n' = 0$ so as to avoid singularities at $r = 0$.

Remarks: The last two sums on the right-hand side of (IV.46) constitute the particular solution w_p of $\nabla^2 w_p = \dfrac{1+\nu}{Kh}\alpha T_d$; more specifically, if

$$
\frac{1+\nu}{hk}\alpha T_d = \begin{cases} A_{km} r^k \cos m\theta \\ B_{km} r^k \sin m\theta \end{cases} \tag{IV.48a}
$$

then $w_p(r,\theta)$ is given by

$$
w_p(r,\theta) = \begin{cases} g_m(r)\cos m\theta \\ h_m(r)\sin m\theta \end{cases} \tag{IV.48b}
$$

with $g_m(r)$, $h_m(r)$ as defined in (IV.47). By carrying out the integrations in (IV.47) and using the results in (IV.48b) it can be shown that

$$
w_p(r_1\theta) = \begin{cases} \dfrac{A_{km} r^{k+2}\cos m\theta}{(k+2)^2 - m^2} \\ \dfrac{B_{km} r^{k+2}\sin m\theta}{(k+2)^2 - m^2} \end{cases} \tag{IV.49a}
$$

when $k+2-m \neq 0$, and (IV.48a) applies, while for the case in which $k+2-m=0$,

$$w_p(r,\theta) = \begin{cases} A_{km}\left\{\dfrac{lnr}{2m} - \dfrac{1}{(2m)^2}\right\} r^m \cos m\theta \\ B_{km}\left\{\dfrac{lnr}{2m} - \dfrac{1}{(2m)^2}\right\} r^m \sin m\theta \end{cases} \tag{IV.49b}$$

Returning to the case of a solid plate, for which $c_n = c'_n = d_n = d'_n = 0$, we note that the constants a_n and b_n must be determined from the boundary conditions. For the case of a clamped plate, in which $w(b,\theta) = \dfrac{\partial w}{\partial r}(b,\theta) = 0$, the (rather complex) expressions for the deflection, moments, and shears in nondimensional form are given in [11]; these results, for $m = 0, 1, 2, 3$ are depicted in Fig. 1.

The problem of bending of a rectangular orthotropic plate (which has two opposite edges simply supported and the other two clamped) due to different temperature distributions on the plate surfaces, has been considered by Misra [12]. It is assumed in [12] that the plate occupies the region

$$0 \le x \le a, \quad -\frac{b}{2} \le y \le \frac{b}{2}, \quad -\frac{h}{2} \le z \le \frac{h}{2}$$

so that the opposite faces are defined by $z = \pm\dfrac{h}{2}$. It is also assumed that the two parallel edges at $x=0$ and $x=a$ are simply supported while the edges $y = \pm\dfrac{b}{2}$ are clamped. The temperature distribution is taken as having the form

$$\delta T(x,y,z) = \frac{T_1+T_2}{2} + \frac{T_1-T_2}{h} z \tag{IV.50}$$

so that $T(x,y,\frac{h}{2}) = T_1$, $T(x,y,-\frac{h}{2}) = T_2$, for $0 \le x \le a$, $-\frac{b}{d} \le y \le \frac{b}{2}$, where T_1 and T_2 are, respectively the constant temperatures at the top and bottom of the plate; thus the temperature is assumed to remain constant in any plane which is parallel to the x, y plane. The edge conditions are given by

$$w = 0, \;\; M_x = 0; \;\; \text{on } x = 0, a, \;\text{for } -\frac{b}{2} \le y \le \frac{b}{2} \tag{IV.51a}$$

and

$$w = 0,\ w_{,y} = 0; \quad \text{on } y = \pm\frac{b}{2}, \text{ for } 0 \le x \le a \tag{IV.51b}$$

where M_x is given by (III.12). Actually, Misra [12] writes the term $\tilde{M}_T^1$ in (III.12) in the form $\tilde{M}_T^1 = \bar{\beta}_1 M_T$, with $M_T = \int_{-h/2}^{h/2} \delta T z dz$, so that (see (III.13))

$$\bar{\beta}_1 = c_{11}\alpha_1 + c_{12}\alpha_2 \tag{IV.52}$$

The superposed bar over the β_1 in (IV.52) does not appear in [12] and has been placed there so as to not confuse this parameter with a hygroscopic coefficient. With the definition of M_T, as given above, and (IV.50) it is easily seen that

$$M_T = \frac{1}{12}h^2(T_1 - T_2) \equiv k \tag{IV.53}$$

which has the Fourier representation

$$M_T = \frac{4k}{\pi} \sum_{m=1,3,5,\ldots}^{\infty} \frac{1}{m} \sin\frac{m\pi}{a}x, \tag{IV.54}$$

for $0 < x < a$. Using (IV.54) in (III.14a), setting $t \equiv 0$ and $\Phi \equiv 0$, and replacing $D_{11} + 4D_{66} + D_{21}$ by $2H$ we obtain, as in [12], the following equation for the bending of the heated, rectangular, orthotropic plate:

$$D_{11}w_{,xxxx} + 2Hw_{,xxyy} + D_{22}w_{,yyyy} = P \sum_{m=1,3,..}^{\infty} m \sin\frac{m\pi}{a}x \tag{IV.55}$$

where

$$P = \frac{4k\pi\bar{\beta}_1}{a^2} \tag{IV.56}$$

From (IV.54) it follows that $M_T = 0$ along the edges at $x = 0$ and $x = a$. Then, by virtue of (IV.51a) it follows that both w and $w_{,yy}$ must vanish along $x = 0$ and $x = a$. However, as $M_x = 0$ along $x = 0$ and $x = a$, it would follow from (III.12) that $w_{,xx} = 0$ along $x = 0$ and $x = a$ only if $M_T = 0$ along these edges, which it does not–the Fourier representation not withstanding! Thus, the edge conditions in (IV.51a), which in [12] are now written in the form

$$w = 0, w_{,xx} = 0, \text{ on } x = 0, a \qquad \text{for } -\frac{b}{2} \leq y \leq \frac{b}{2} \tag{IV.57}$$

are open to suspicion as is, of course, the remainder of the solution presented below. A solution of the homogeneous equation associated with (IV.55) which is compatible with the edge conditions (IV.57) is now sought in [12] in the form

$$w = \sum_{m=1,3,..}^{\infty} Y_m(y) \sin \frac{m\pi}{a} x \tag{IV.58}$$

Substituting this expansion into the relevant homogeneous partial differential equation we are led to the following homogeneous fourth order ordinary differential equation for the functions $Y_m(y)$:

$$D_{22}Y_m'''' - 2H\alpha_m^2 Y_m'' + D_{11}\alpha_m^4 Y_m = 0 \tag{IV.59}$$

where $\alpha_m = \dfrac{m\pi}{a}$. Noting that, because of symmetry, Y_m must be an even function of y, a solution of (IV.59) is sought, in [12], in the form

$$Y_m(y) = A_m \cosh\ p_m y \cos q_m y + B_m \sinh\ p_m y \sin q_m y \tag{IV.60}$$

where

$$\begin{cases} p_m^2 = \dfrac{\alpha_m^2(H + \sqrt{H^2 - D_{11}D_{22}})}{D_{22}} \\ q_m^2 = \dfrac{\alpha_m^2(H - \sqrt{H^2 - D_{11}D_{22}}}{D_{22}} \end{cases} \tag{IV.61}$$

The A_m, B_m are, at this junction, arbitrary functions of m. For a particular integral of (IV.55) Misra [12] chooses

$$w = \sum_{m=1,3,5...}^{\infty} E_m \sin \alpha_m x \tag{IV.62}$$

Substitution of (IV.62) into (IV.55) then yields

$$E_m = \frac{mP}{\alpha_m^4 D_{11}} \tag{IV.63}$$

in which case the complete solution of (IV.55) assumes the form

$$w = \sum_{m=1,3,5...}^{\infty} \{\frac{mP}{\alpha_m^4 D_{11}} + A_m \cos h p_m y \cos q_m y + B_m \sinh\ p_m y \sin q_m y\} \sin \alpha_m x \qquad \text{(IV.64)}$$

The edge conditions (IV.57) are automatically satisfied by (IV.64) while those in (IV.51b) are satisfied if and only if A_m, B_m are connected by the relations

$$\frac{mp}{\alpha_m^4 D_{11}} + A_m \cosh \frac{bpm}{2} \cos \frac{bqm}{2} + B_m \sinh\ \frac{bpm}{2} \sin \frac{bqm}{2} = 0 \qquad \text{(IV.65a)}$$

$$A_m(p_m \tan h\frac{bpm}{2} - q_m \tan \frac{bqm}{2}) + B_m(p_m \tan \frac{bqm}{2} + q_m \tanh \frac{bpm}{2}) = 0 \qquad \text{(IV.65b)}$$

These relations may be solved for A_m, B_m (see [12] for the details) which are then substituted back into (IV.64) so as to yield the deflection at any point of the plate. The expression obtained for $w(x, y)$ can also be employed in (III.12) so as to compute the thermally induced moments at any point in the plate, e.g.,

$$\begin{aligned}
M_x = D_{11} \sum_{m=1,3,5,..}^{\infty} \alpha_m^2 \{A_m \cosh p_m y \cos q_m y \\
+B_m \sinh p_m y \sin q_m y\} \sin \alpha_m x \\
-D_{12} \sum_{m=1,3,5,...}^{\infty} [A_m\{(p_m^2 - q_m^2) \cosh p_m y \cos q_m y \\
-2p_m q_m \sinh p_m y \sin q_m y\} \\
+B_m\{(p_m^2 - q_m^2) \sinh p_m y \sin q_m y \\
+2p_m q_m \cosh p_m y \cos q_m y\}] \sin \alpha_m x \\
-\frac{4\bar{\beta}_1 k}{a} \sum_{m=1,3,5,...}^{\infty} \frac{1}{\alpha_m} \sin \alpha_m x
\end{aligned} \qquad \text{(IV.66)}$$

with analogous expressions for M_y and M_{xy}. Finally, the deflection at the center of the plate, i.e., at $x = \frac{a}{2}, y = 0$, is computed in [12] to be

$$w = \sum_{m=1,3,5\ldots}^{\infty} \left(\frac{mP}{\alpha_m^4 D_{11}} + A_m\right) \sin \frac{m\pi}{2} \tag{IV.67}$$

The expressions for the moments, e.g., (IV.66) and the final result (IV.67) for the deflection at the center of the plate are subject to the criticism (of the relevance of the boundary conditions (IV.57)) which has been levied above.

An alternative approach to solving problems of thermal deflection for plates that has been used extensively in the literature is based upon the concept of an influence function and usually goes under the title of Maysel's method; this approach is actually an extension of Betti's reciprocal theorem to thermoelastic problems and excellent treatments have appeared in several places in the literature, e.g., in Nowacki [8] and in Tauchert [3]. In what follows we will adhere closely to the presentation in [3] and will assume that the plate exhibits isotropic response; we will also take $t \equiv 0$, so that in either rectangular or polar coordinates the relevant partial differential equation is given by

$$K\triangle^2 w = -\triangle M^T \tag{IV.68}$$

If the plate occupies the domain $\mathcal{A}$ in the x, y plane, when in its undeflected state, and $w^*(\xi, y; x, y)$ is the Green's function for the operator $K\triangle^2$ then it is easily shown (i.e. [3] or [8]) that

$$w(x,y) = -\iint_{\mathcal{A}} M^T(\xi,\eta) \, \nabla^2 \, w^*(\xi,\eta;x,y) d\mathcal{A}(\xi,\eta) \tag{IV.69}$$

where, for the sake of convenience, we have initiated the discussion by employing rectangular coordinates. In (IV.69),

$$\nabla^2 = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}$$

The Green's function in (IV.69), $w^*(\xi,\eta;x,y)$, represents the deflection at the point (ξ,η) of the plate middle surface which would be due to a concentrated unit force applied at the point (x, y). Thus, the Maysel relation (IV.69) may be used to compute a thermally induced deflection $w(x, y)$ whenever $w^*(\xi, \eta; x, y)$ can be calculated for a plate of given shape and assigned support conditions. As an alternative to (IV.69) one may use the form obtained by employing Green's formula, i.e.,

$$\begin{aligned} w(x,y) &= -\iint_{\mathcal{A}} w^*(\xi,\eta;x,y)\, \nabla^2 M^T(\xi,\eta) d\mathcal{A}(\xi,\eta) \\ &\quad - \int_{\partial\mathcal{A}} \left(M^T \frac{\partial w^*}{\partial n} - w^* \frac{\partial M^T}{\partial n} \right) ds \end{aligned} \tag{IV.70}$$

where n, s denote, respectively, the directions that are normal and tangential to the plate boundary $\partial\mathcal{A}$. If $\nabla^2 M^T = 0$, such as for the case in which M^T is constant, then (IV.70) reduces to

$$w(x,y) = -\int_{\partial\mathcal{A}} \left(M^T \frac{\partial w^*}{\partial n} - w^* \frac{\partial M^T}{\partial n} \right) ds \tag{IV.71}$$

and if the plate is simply supported, so that $w^* = 0$ along $\partial\mathcal{A}$, then

$$w(x,y) = -\int_{\partial\mathcal{A}} M^T(\xi,\eta) \frac{\partial w^*}{\partial n}(\xi,\eta;x,y) ds \tag{IV.72}$$

Of course, if $\nabla^2 M^T = 0$ in $\mathcal{A}$, and the plate is clamped along $\partial\mathcal{A}$, then $w^* = \dfrac{\partial w^*}{\partial n} = 0$ along $\partial\mathcal{A}$ in which case $w \equiv 0$ throughout the plate.

As a first example of the influence function method we consider the simply supported rectangular plate which is depicted in Fig. 2. We assume that the thermal moment is nonzero within an arbitrary region $\mathcal{A}^T$ of the plate while $M^T = 0$ in the complement of this region. It is easily shown that the deflection w^*, at an arbitrary point (ξ, η) of the same simply supported plate subject to a concentrated unit force at (x, y) is

$$w^*(\xi,\eta;x,y) = \frac{4}{abK} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin \alpha_m \xi \sin \beta_n \eta \sin \alpha_m x \sin \beta_n y}{(\alpha_m^2 + \beta_n^2)^2} \tag{IV.73}$$

where $\alpha_m = \dfrac{m\pi}{a}$, $\beta_n = \dfrac{n\pi}{b}$ provided M^T vanishes along the edges of the plate. Substituting (IV.73) into (IV.69), and carrying out an elementary computation, we are led to the following expression for the deflection:

$$w(x,y) = \frac{4}{abK}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\sin\alpha_m x\sin\beta_n y}{(\alpha_m^2+\beta_n^2)^2}\iint_{A^T} M^T(\xi,\eta)\sin\alpha_m\xi\sin\beta_n y d\xi d\eta \tag{IV.74}$$

If the thermal moment is constant over the entire plate, say, $M^T = M$, then (IV.74) formally reduces to

$$w(x,y) = \frac{16M}{abK}\sum_{m=1,3,\ldots}^{\infty}\sum_{n=1,3,\ldots}^{\infty}\frac{\sin\alpha_m x\sin\beta_n y}{\alpha_m\beta_n(\alpha_m^2+\beta_n^2)} \tag{IV.75}$$

but, once again, such a solution is subject to the criticisms raised earlier as, now, M^T does not vanish along the edges of the plate.

As a second example, we consider the application of Maysel's relation (IV.69) to the thermal deflection of a solid circular plate of radius b; in this case (IV.69) assumes the following form in terms of polar coordinates:

$$w(r,\theta) = -\int_0^{2\pi}\int_0^b M^T(\rho,\psi)\,\nabla^2\, w^*(\rho,\psi;N,\theta)\rho d\rho d\psi \tag{IV.76}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}+\frac{1}{\rho^2}\frac{\partial^2}{\partial\psi^2}$$

In lieu of (IV.76) we may write, in analogy with (IV.70), that

$$\begin{aligned} w(r,\theta) = &-\int_0^{2\pi}\int_0^b w^*(\rho,\psi;r,\theta)\,\nabla^2\, M^T(\rho,\psi)\rho d\rho d\psi \\ &-\int_0^{2\pi}\left(M^T(b,\psi)\frac{\partial w^*(b,\psi;r,\theta)}{\partial\rho} - w^*(b,\psi;r,\theta)\frac{\partial M^T(b,\psi)}{\partial\rho}\right) bd\psi \end{aligned} \tag{IV.77}$$

If the circular plate is clamped along its edge at $r = b$ then $w^* = \dfrac{\partial w^*}{\partial\rho} = 0$, for $r = b$, $0 \le \theta < 2\pi$, and (IV.77) reduces to

$$w(r,\theta) = -\int_0^{2\pi}\int_0^b w^*(\rho,\psi;r,\theta)\,\nabla^2\, M^T(\rho,\psi)\rho d\rho d\psi \tag{IV.78}$$

The appropriate Green's function w^* in (IV.78) for the case of a clamped edge is known to have the form

$$w^*(\rho,\psi;r,\theta) = \frac{b^2}{16\pi K}\Bigg\{(1-\rho'^2)(1-r'^2)$$
$$\left(\rho'^2 + r'^2 - 2\rho' r'\cos(\theta-\psi) ln\frac{\rho'^2 + r'^2 - 2\rho' r'\cos(\theta-\psi)}{1+\rho'^2 r'^2 - 2\rho' r'\cos(\theta-\psi)}\right)\Bigg\} \tag{IV.79}$$

with $\rho' = \rho/b$ and $r' = r/b$. Various forms of the Green's function are available for the case of the simply supported solid circular isotropic plate but, as noted in [3] the expressions tend to be quite complex.

Before concluding this brief description of thermal bending of plates (and moving on to describe some problems of thermal buckling, within the context of small deflection theory) we want to note some approximate methods that have been employed to deal with problems of thermal flexure of plates; two of the better-known techniques are the Rayleigh-Ritz and Galerkin procedures. In the Rayleigh-Ritz method the displacement field w is approximated by functions which contain a finite number of independent coefficients. The functions employed are chosen so as to satisfy the kinematic boundary conditions but they do not have to satisfy the static boundary conditions. The unknown coefficients in the assumed solution are then determined by employing the principle of minimum potential energy. For the problem of thermal flexure we may, in particular, represent the transverse displacement $w(x,y)$ in the form

$$w(x,y) = \sum_{m=1}^{M}\sum_{n=1}^{N} c_{mn}\phi_{mn}(x,y) \tag{IV.80}$$

It is assumed here that the $\phi_{mn}(x,y)$ satisfy the boundary conditions which involve w, $w_{,x}$, and $w_{,y}$. The assumed form of the solution (IV.80) is then substituted into the expression for the potential energy Π which, for a problem of (purely) thermal flexure of a homogeneous isotropic plate, is given by the following reduced form of (III.41):

$$\begin{aligned}\Pi = \int_0^b\int_0^a \Bigg\{ & \frac{K}{2}(w_{,xx} + w_{,yy})^2 \\ & +(1-\nu)K(w_{,xy}^2 - w_{,xx}w_{,yy}) \\ & +M^T(w_{,xx} + w_{,yy}) - tw\Bigg\}dxdy\end{aligned} \tag{IV.81}$$

Setting $\delta\Pi = 0$, after substituting (IV.80) into (IV.81), yields a system of $M+N$ simultaneous algebraic equations, i.e.,

$$\frac{\partial \Pi}{\partial c_{mm}} = 0; \; m = 1, 2, ..., M; \; n = 1, 2, ..., N \tag{IV.82}$$

which are then employed so as to compute the c_{mn}. To illustrate the use of the Rayleigh-Ritz procedure, we may consider the simple example of a square plate of side length a which is simply supported along the edges at $x = 0$ and $x = a$, clamped along the edges at $y = 0$ and $y = a$, and subjected to a uniform thermal moment M^T. If we use the representation

$$w = \sum_{m=1}^{M} \sum_{n=1}^{N} c_{mn} \sin \frac{m\pi x}{a} \left(1 - \cos \frac{2n\pi y}{a}\right) \tag{IV.83}$$

for the transverse deflection, then we satisfy the kinematic boundary conditions for this problem but not the static boundary condition $M_x = 0$ along the edges $x = 0$ and $x = a$, $0 \leq y \leq a$. Retaining only the term corresponding to $m = 1$, $n = 1$ in (IV.83) it is easily verified that the Rayleigh-Ritz method yields an approximation to $w(x, y)$ in which the maximum deflection, which occurs at $x = y = \frac{1}{2}a$, is given by $0.0191a^2M^T/K$. As noted in [3], two and three term approximations, using (IV.83), yield maximum deflections of $0.0144a^2M^T/K$ and $0.0157a^2M^T/k$, respectively, while the 'exact' value of the maximum deflection in this case is given (approximately) by $0.0158a^2M^T/K$.

To implement the Galerkin procedure, we work directly with the relevant differential equation instead of with the associated potential energy; the equation, for the problem of thermal flexure of an isotropic, homogeneous thin plate is just (III.4a) with $\Phi \equiv 0$, i.e.,

$$K\Delta^2 w + \Delta M^T - t = 0 \tag{IV.84}$$

An approximate solution of the form (IV.80) is again sought, the difference being that the $\phi_{mn}(x, y)$ must satisfy all the pertinent boundary conditions. If we substitute (IV.80) into (IV.84) we will obtain an error (or residual) $e(x, y)$ which is given by

$$e(x, y) = K\Delta^2 w + \nabla^2 M^T - t \tag{IV.85}$$

and in the Galerkin method it is required that $e(x, y)$ be orthogonal to each of the $\phi_{mn}(x, y)$, i.e., that (assume a rectangular plate, $0 \leq x \leq a$, $0 \leq y \leq b$)

$$\left\{ \begin{array}{c} \displaystyle\int_0^b \int_0^a e(x,y)\phi_{mn}(x,y)dxdy = 0, \\ m = 1, 2, ..., M \\ n = 1, 2, ...N \end{array} \right. \tag{IV.86}$$

By computing the integrals in (IV.86) we are led to a system of $M + N$ algebraic equations for the coefficients c_{mn}.

Remarks: If one incorporates boundary residuals into the Galerkin procedure it is possible to relax the constraint that the $\phi_{mn}(x, y)$ satisfy the static as well as the kinematic boundary conditions. The first variation $\delta\Pi$ of the total potential energy Π is, for purely thermal flexure problems, given by the following reduced (and modified) form of (III.42)

$$\begin{aligned} \delta\Pi \;=\; & \int_0^b \int_0^a (K\Delta^2 w + \Delta M^T - t)\delta w dxdy \qquad\qquad \text{(IV.87)} \\ & + \int_0^b \left\{ \left[(M_{x,x} + 2M_{xy,y} - \bar{K}_x)\delta w\right]_{x=0}^{x=a} \right. \\ & \qquad \left. - \left[(M_x - \bar{M}_x)\delta\left(\frac{\partial w}{\partial x}\right)\right]_{x=0}^{x=a} \right\} dy \\ & + \int_0^a \left\{ \left[(M_{y,y} + 2M_{xy,x} - \bar{K}_y)\delta w\right]_{y=0}^{y=b} \right. \\ & \qquad \left. - \left[(M_y - \bar{M}_y)\delta\left(\frac{\partial w}{\partial w}\right)\right]_{y=0}^{y=b} \right\} dx \\ & - \left[(2M_{xy} - \bar{R}_{xy})\delta w\right]_{x=0,y=b}^{x=a,y=b} \\ & + \left[(2M_{xy} - \bar{R}_{xy})\delta w\right]_{x=0,y=0}^{x=a,y=0} = 0 \end{aligned}$$

Equation (IV.87) includes the possibility of nonzero prescribed edge and corner loads $\bar{K}_x$, $\bar{K}_y$, $\bar{M}_x$,

$\bar{M}_y$, and $\bar{R}_{xy}$. The variation δw is, by virtue of (IV.80), computed as

$$\delta w = \sum_{m=1}^{M} \sum_{n=1}^{N} \delta c_{mn} \phi_{mn}(x, y) \tag{IV.88}$$

If all of the boundary conditions are of kinematic type, then substitution of (IV.80) and (IV.88) into (IV.87) yields

$$\begin{aligned} &\int_0^b \int_0^a (K\Delta^2 w + \Delta M^T - t)\phi_{mn}(x, y)dxdy = 0 \\ &(m = 1, 2, ..., M; n = 1, 2, ..., N) \end{aligned} \tag{IV.89}$$

which is, of course, equivalent to (IV.85). One also obtains (IV.89) if certain of the boundary conditions, as noted in [3], are static; however, these static conditions must be satisfied identically by (IV.80). Suppose we consider, as an example, the case treated earlier in this section by the Raleigh-Ritz method, i.e., a square plate clamped along two parallel edges and simply supported along the other two, and subjected to a uniform thermal moment M^T. For this problem, the static boundary condition $M_x = 0$ is not satisfied, as already noted, by the assumed form (IV.83) of the solution. The condition (IV.87) leads, in this case to the following system of equations for the coefficients c_{mn}:

$$\begin{aligned} &\int_0^b \int_0^a K\Delta^2 w \cdot \phi_{mn}(x, y)dxdy \\ &\qquad + \int_0^b \left[(Kw_{,xx} + \nu K w_{,yy} + M^T)\phi_{mn,x}\right]_{x=0}^{x=a} dy = 0 \\ &\qquad\qquad (m = 1, 2, ..., M;\; n = 1, 2, ..., N) \end{aligned} \tag{IV.90}$$

It is easily demonstrated that the coefficients c_{mn} which are determined by solving (IV.90) are, in fact, identical to those that are obtained by applying the Rayleigh-Ritz procedure.

In all of the work discussed, to this point, in this section not only have we assumed that we are working within the domain of small deflection theory but also, that the stress resultants in the plane of the plate were small enough so as to not materially influence the transverse deformations of the plate; if such is not the case then, e.g., for an isotropic, thin, elastic plate in rectangular coordinates, the basic equations governing the flexure and buckling of the plate are (see (III.4a,b))

$$K\Delta^2 w = t - \Delta M^T + N_x w_{,xx} + N_y w_{,yy} + 2N_{xy} w_{,xy} \tag{IV.91a}$$

$$\Delta^2 \Phi = -(1-\nu)\Delta N^T \tag{IV.91b}$$

where Φ is given by (I.20), N^T by (I.11a), M^T by (I.14), and the small deflection assumption has been enforced in writing down (IV.91b). For a given temperature distribution $\delta T(x,y,z)$, and given boundary conditions along the edge of the plate, one would first compute ΔN^T and then solve (IV.91b) for $\Phi \equiv \Phi_0(x,y)$; the airy function Φ_0 is then used to compute the in-plane, pre-buckling stress resultants N_x^0, N_y^0, N_{xy}^0, which are substituted into (IV.91a), along with t and ΔM^T. Finally (IV.91a), together with appropriate support conditions with respect to w along the edges of the plate, is treated as an eigenvalue-eigenfunction problem with the first eigenvalue (for a purely thermal problem) corresponding to the (smallest) critical temperature and the corresponding eigenfunction representing the first buckling mode. In order to illustrate the procedure delineated above, we will begin our discussion by presenting three examples that have been highlighted in Boley and Weiner [7] for isotropic plates and a rectilinear geometry; we will then proceed to examples involving circular plates as well as problems for plates with orthotropic material symmetry.

The first case treated in [7] concerns the buckling of plates subjected to heat conduction (but no transverse loads) with their edges unrestrained in the plane. We are reminded in [7] of the basic fact that if the ends of a column are free to displace axially, and the column is free from axial loads, then the column can not buckle no matter what the temperature distribution may be; this is clearly not the case with plates. Because the plate is assumed to be free of external tractions in its plane, equilibrium relations of the form

$$\int N_x dy = 0 \tag{IV.92}$$

have to be satisfied in which the integration extends across the entire plate along a line given by $x =$ const; a relation such as (IV.92) can not hold unless $N_x > 0$ along part of this line while $N_x < 0$ along its complement thus leading to the conclusion that for this class of problems compressive stresses will always occur in the plane of the plate. A very well-known example of the type referenced above occurs in the often quoted paper of Gossard, Seide, and Roberts [2] which will be discussed in some detail in §V; although the focus, in §V, with respect to the discussion of the work in [2], will be on postbuckling behavior, it should be clear that the buckling problem described, e.g., by the system (III.4a,b), within the context of small deflection theory is, mathematically isomorphic to the initial buckling problem for the full non-linear system. Indeed, some specific initial buckling problems for such systems will be discussed at the end of this section.

A second class of thermal buckling problems, in the realm of small deflection theory, which is discussed in [7] and which is mathematically similar to the first class of problems, concerns the buckling of plates which are subjected to heat and loads in their plane with, once again, their edges unrestrained in the plane of the plate. As an example, we consider the plate strip of Fig. 3 which is loaded at its ends by a uniformly distributed stress σ_0; the strip, of width b, is reinforced along its edges at $y = 0$, $y = b$ by longitudinals of area A which act as a heat sink, thus, causing the temperature to be higher along the center of the plate than near its edges. For illustration purposes the temperature will be assumed to be uniform across the thickness of the plate and of the form

$$\delta T(x, y) = c_0 - c_1 \cos\left(\frac{2\pi y}{b}\right) \tag{IV.93}$$

in the plane of the plate where c_0, c_1 are constants which may be chosen so as to fit empirical data. We consider a single panel of the strip, as depicted in Fig. 3., which extends from $x = 0$ to $x = a$; it is assumed that this panel is at a large enough distance from the ends of the strip so that the stresses can be taken to be independent of x. Also, we assume that $w = 0$ along the line segments $x = 0, x = a$, for $0 \leq y \leq b$. As we have already indicated in the discussion of the procedure for solving (IV.91a,b), the first step in the solution of the problem at hand consists of determining a stress function Φ from (IV.91b) and the pertinent boundary conditions; these boundary conditions are as follows:

$$\begin{cases} u(0,y)=0, \; u(a,y)=u_0, \; 0\le y\le b \\ v(x,0)=0, \; v(x,b)=v_0, \; 0\le x\le a \end{cases} \tag{IV.94}$$

where u_0 and v_0 are constants which are chosen so that

$$\int_0^b N_x(0,y)dy = \int_0^b N_x(a,y)dy = bh\sigma_0 \tag{IV.95}$$

For the temperature distribution (IV.93), (I.11a) and (IV.91b) yield.

$$\Delta^2\Phi = -4\left(\frac{\pi}{b}\right)^2 \alpha E c_1 \cos\frac{2\pi y}{b} \tag{IV.96}$$

a solution of which is

$$\Phi \equiv \Phi_0(x,y) = \frac{\sigma_0 y^2}{2} - \frac{\alpha E c_1 b^2}{4\pi^2}\cos\frac{2\pi y}{b} \tag{IV.97}$$

It is easily computed that corresponding to Φ_0, as given by (IV.97), we have the following expressions, modulo rigid-body motions, for the stress, strain, and displacement components:

$$\begin{cases} \sigma_{yy} = N_y = \sigma_{xy} = N_{xy} = \epsilon_{xy} = 0 \\ h\sigma_{xx} = N_x = \Phi_{,yy} = h\left\{\sigma_0 + \alpha E c_1 \cos\frac{2\pi y}{b}\right\} \end{cases} \tag{IV.98a}$$

$$\begin{cases} \epsilon_{xx} = \frac{1}{E}\sigma_0 + \alpha c_0 \\ \epsilon_{yy} = -\frac{\nu\sigma_0}{E} + \alpha c_0 - (1+\nu)\alpha c_1 \cos\frac{2\pi y}{b} \end{cases} \tag{IV.98b}$$

$$\begin{cases} u = (\sigma_0 + \alpha E c_0)\frac{x}{E} \\ v = (-\nu\sigma_0 + \alpha E c_0)\frac{y}{E} - \frac{(1+\nu)\alpha c_1 b}{2\pi}\sin\frac{2\pi y}{b} \end{cases} \tag{IV.98c}$$

It is easily seen that (IV.98a,b,c) satisfy all the boundary conditions delineated above provided

$$\begin{cases} u_0 = (\sigma_0 + \alpha E c_0)\frac{a}{E} \\ v_0 = (-\nu\sigma_0 + \alpha E c_0)\frac{b}{E} \end{cases} \tag{IV.99}$$

The next step, for the problem at hand, involves the computation of the transverse deflection $w(x, y)$ and the corresponding critical combination of the temperature levels and the applied load. Using the fact that $t \equiv 0$ in (IV.91a), and that $M^T = 0$ for the temperature distribution defined by (IV.9b), it is easily seen that the use of (IV.98a) in (IV.91a) reduces this equation to

$$\Delta^2 w = \frac{h}{K}\left\{\sigma_0 + \alpha E c_1 \frac{\cos 2\pi y}{b}\right\} w_{,xx} \tag{IV.100}$$

We consider (IV.100) with the conditions relevant for simply supported edges, namely,

$$\begin{cases} w = w_{,xx} = 0; \; x = 0, a; \; 0 \le y \le b \\ w = w_{,yy} = 0; \; y = 0, b; \; 0 \le x \le a \end{cases} \tag{IV.101}$$

and, thus, seek a solution of the form

$$w(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \tag{IV.102}$$

By substituting (IV.102) into (IV.100) and then comparing the coefficients of like terms we are led to the following system of algebraic equations for the coefficients a_{mn}:

$$\begin{cases} \left[k_{m1} + \sigma_0 - \dfrac{\alpha E c_1}{2}\right] a_{m1} + \dfrac{\alpha E c_1}{2} a_{m3} = 0 \\[2ex] \left[k_{m2} + \sigma_0\right] a_{m2} + \dfrac{\alpha E c_1}{2} a_{m4} = 0 \\[2ex] \left[k_{mn} + \sigma_0\right] a_{mn} + \dfrac{\alpha E c_1}{2}(a_{m,n+2} + a_{m,n-2}) = 0, \; (n > 2) \end{cases} \tag{IV.103}$$

where

$$k_{mn} = \frac{K}{h}\left(\frac{m\pi}{a}\right)^2 \left\{1 + \left(\frac{na}{mb}\right)^2\right\}^2 \tag{IV.104}$$

The critical combination of σ_0 and c_1 is obtained by setting the determinant of the homogeneous system (IV.103) equal to zero. As noted in [7] there does not exist any coupling between coefficients with different values of m or between coefficients with even and odd values of n. Thus, a single value of m may be employed in the series (IV.102), i.e., the one which yields the lowest critical combination of load and temperature levels for the loading and geometry being considered. Furthermore, as also noted in [7], one may set up two independent determinants, one with only odd values of n and one with even values only. It may be shown that the symmetric case, corresponding to the determinant involving only odd values of n, corresponds to the lower 'buckling' load; the symmetric determinant has the form

$$\begin{vmatrix} \{k_{m1}+\sigma_o-\frac{\alpha E c_1}{2}\} & \frac{\alpha E c_1}{2} & 0\cdots \\ \frac{\alpha E c_1}{2} & k_{m3}+\sigma_o & \frac{\alpha E c_1}{2}\cdots \\ 0 & \frac{\alpha E c_1}{2} & (k_{m5}+\sigma_o)\cdots \\ \cdots & \cdots & \cdots \end{vmatrix} = 0 \qquad \text{(IV.105)}$$

for which two special cases are of interest: If $c_1 = 0$ then only the edge stress distribution σ_o acts to buckle the panel and only the diagonal terms in (IV.105) survive. In this case, the critical value of σ_o is given by the same expression that has already been noted in [1], namely,

$$(\sigma_o)_{cr}|_{c_1=0} = -K\frac{\pi^2 E}{12(1-\nu^2)}(\frac{h}{b})^2 \qquad \text{(IV.106a)}$$

in which $n = 1$ (so as to obtain the lowest possible critical stress) and

$$k = (\frac{bm}{a} + \frac{a}{bm})^2 \qquad \text{(IV.106b)}$$

is computed, for a given aspect ratio a/b, by choosing the integral value of m for which it is a minimum; a full discussion may be found in [1]. The more interesting special case of (IV.105)), from the viewpoint of (purely) thermal buckling, corresponds to taking $\sigma_o = 0$ in (IV.105) and seeking the smallest root $T_1 = T_{cr}$ of the resulting infinite determinant; approximations to T_{cr} may be obtained from (IV.105), with $\sigma_o = 0$, by retaining only a finite number of rows and columns of the determinant. By retaining only the element in the first row and the first column of (IV.105), and setting $\sigma_o = 0$, we obtain

$$(\frac{\alpha E}{2})T_{cr}|_{\sigma_0=0} \approx k\frac{\pi^2 E}{12(1-\nu^2)}(\frac{h}{b})^2 = -(\sigma_0)_{cr}|_{c_1=0} \tag{IV.107}$$

with k again given by (IV.106b). If the first 2×2 block in (IV.105) is retained, and σ_0 is set equal to zero, it is possible to show that

$$(\frac{\alpha E}{2})T_{cr}|_{\sigma_0=0} = k_1\frac{\pi^2 E}{12(1-\nu^2)}(\frac{h}{b})^2 \tag{IV.108}$$

with the coefficient k_1 given by

$$\begin{aligned} k_1 = \frac{1}{2}(\frac{a}{mb})^2 \Bigg\{ & \sqrt{[(\frac{mb}{a})^2+9]^4 + 4[(\frac{mb}{a})^2+1]^2[(\frac{mb}{a})^2+9]^2} \\ & -[(\frac{mb}{a})^2+9]^2 \Bigg\} \end{aligned} \tag{IV.109}$$

As indicated in [7], computations performed using larger subdeterminants of (IV.105), with $\sigma_o = 0$, yield results which are very close to those presented above. In Fig. 4 we show a plot of k_1 versus the aspect ratio a/b for various values of m; the graphs indicate that for $\frac{a}{b} >> 1$ a good approximation to k_1 is given by $k_1 \approx 3.848$, which is the value that corresponds to $m = \frac{a}{b}$, while for $\frac{a}{b} < 1$ the curve for $m = 1$ in Fig. 4 should be used. The more general case in which both heat and applied edge loads act on the panel can be treated in a manner similar to that for the case in which $\sigma_o = 0$. Retention of the first 2×2 block in (IV.105), with $\sigma_0 \neq 0, c_1 \neq 0$, leads to the results depicted in Fig. 5 which are interpolated quite accurately by the equation

$$\frac{T_{cr}}{T_{cr}|_{\sigma_o=0}} + \frac{(\sigma_o)_{cr}}{(\sigma_o)_{cr}|_{c_1=0}} = 1 \tag{IV.110}$$

for all combinations of heat and edge stress and all aspect ratios.

The last basic example, within the context of small deflection theory, that is presented in [7], concerns plates whose edges are restrained in the plane of the plate. Consider, e.g., the case of a simply supported rectangular plate whose edges are fixed in the plane of the plate and which is subjected to a temperature distribution which varies through the thickness of the plate, i.e. $\delta T = \delta T(z)$, in such a way as to cause bending; no external loads act on the plate. The displacement boundary conditions, when the plate occupies the domain $0 \leq x \leq a, 0 \leq y \leq b, -\frac{h}{2} \leq z \leq \frac{h}{2}$, are

$$u(0,y) = u(a,y) = v(x,0) = v(x,b) = 0 \tag{IV.111}$$

for $0 \leq y \leq b$ and $0 \leq x \leq a$. Under these conditions the solution for the displacement components in the plane of the plate is $u = v = 0$ which implies that (see (III.2))

$$N_x = N_y = -N^T; N_{xy} = 0 \tag{IV.112}$$

so that the in-plane equilibrium equations are automatically satisfied. As $t \equiv 0$, and $\triangle M^T = 0$, equation (III.4a) reduces, in view of (IV.112), to

$$K\triangle^2 w + N^T \triangle w = 0 \tag{IV.113}$$

If we associate with (IV.113) the boundary conditions corresponding to simply supported edges along $x = 0, x = a$ and $y = 0, y = b$ then the pertinent boundary value problem can be shown to be equivalent to

$$\begin{cases} K \nabla^2 w + N^T w = -M^T; 0 < x < a, 0 < y < b \\ w = 0; \begin{cases} x = 0, a; & 0 \leq y \leq b \\ y = 0, b; & 0 \leq x \leq a \end{cases} \end{cases} \tag{IV.114}$$

Taking $w(x,y)$ is the form

$$w = \sum_{m=1}^{\infty} y_m(y) \sin(\frac{m\pi x}{a}) \tag{IV.115}$$

and expanding the constant M^T as

$$M^T = (\frac{4M^T}{\pi}) \sum_{m=1,3,5...}^{\infty} (\frac{1}{m}) \sin(\frac{m\pi x}{a}) \tag{IV.116}$$

the authors [7] obtain the following equation for the y_m:

$$\begin{cases} \frac{d^2 y_m}{dy^2} - \beta_m^2 y_m = -\frac{4M^T}{km\pi}; m = 1,3,5... \\ \beta_m = \sqrt{(\frac{m\pi}{a})^2 - \frac{1}{K} N^T} \end{cases} \tag{IV.117}$$

Once again, as M^T does not vanish along the edges of the plate such a solution technique is subject to the criticisms raised earlier. In Fig. 6 we show a nondimensional plot of the deflection (as computed in [7] at the center of the rectangular plate) against the temperature parameter $N^T/(N^T)_{cr}$ for various aspect ratios $\frac{a}{b}$; here $(N^T)_{cr}$, the value of N^T at which buckling occurs is computed to be

$$(N^T)_{cr} = (1 + \frac{a^2}{b^2})\frac{\pi^2}{a^2} \tag{IV.118}$$

The (nondimensional) variation of the bending moment M_x in a square plate is shown in Figs. 7 and 8 for two different values of the 'temperature level' $N^T/(N^T)_{cr}$; such plots are sufficient to determine both M_x and M_y throughout the entire plate because of the double symmetry exhibited by the plates; the results depicted in Figs. 7 and 8 indicate that the maximum bending moment occurs at the center of the plate. Finally, in Fig. 9 we show, for various aspect ratios, the variations in M_x, at the center of the plate, with the temperature parameter $N^T/(N^T)_{cr}$. All of the results depicted in Figs. 6-9 are valid only for values of N^T that are sufficiently close to $(N^T)_{cr}$ because of the small-deflection assumption employed in their derivation.

For the special case (in rectangular coordinates) in which (III.5) reduces to

$$\delta T(x,y,z) = T_o(x,y) \tag{IV.119}$$

so that, by virtue of (III.6),

$$N^T = \frac{\alpha E h}{1-\nu} T_o(x,y); \;\; M^T = 0 \tag{IV.120}$$

Equations (III.4a) and (III.46) for the case of isotropic response, reduce to

$$K\triangle^2 w = \Phi_{,yy} w_{,xx} - 2\Phi_{,xy} w_{,xy} + \Phi_{,xx} w_{,yy} \tag{IV.121a}$$

$$\triangle^2 \Phi + \alpha E h \triangle T_o = 0 \tag{IV.121b}$$

provided $t \equiv 0$ and small-deflection theory is assumed. Equations (IV.121a, b) appear, e.g., in Nowacki [8] with $F = \frac{1}{h}\Phi$, the Airy function associated with the stress components, in lieu of Φ. If we assume that, for a given temperature field (IV.119), equation (IV.121b) has been solved for Φ, then the stress resultants N_x, N_y, N_{xy} are known and equation (IV.121a), namely,

$$\triangle^2 w = \frac{1}{K}(N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy}) \equiv \frac{1}{K} N_{ij} w_{,ij} \tag{IV.122}$$

must be solved subject to the specification of boundary conditions on w. As an example we consider a rectangular plate with $0 \leq x \leq a, 0 \leq y \leq b$, which is simply supported along its edges; for this case $M^T = 0$ along the edges of the plate so our previous criticisms do not apply.

One approach to dealing with (IV.22) is similar to Maysel's method which was discussed in connection with the thermal bending of plates. We introduce the Green's function w^* which satisfies the equation

$$\triangle w^*(x, y; \xi_1, \xi_2) = \frac{1}{K}\delta(x - \xi_1)\delta(y - \xi_2) \tag{IV.123}$$

in the sense of distributions and the same boundary conditions as w. Combining (IV.122) and (IV.123) we obtain

$$w(x, y) = \int_o^a \int_o^b w^*(x, y; \xi_1, \xi_2) N_{ij}(\xi_1, \xi_2) w_{,ij}(\xi_1, \xi_2) d\xi_1 d\xi_2 \tag{IV.124}$$

If we apply the Green's transformation to the right-hand side of (IV.124), assume that the plate is simply supported along its edges (or clamped), and use the planar equilibrium equations $N_{ij,ij} = 0$, we find that (IV.24) yields the following Fredholm integral equation of the second kind for $w(x, y)$:

$$w(x, y) = \int_o^a \int_o^b w(\xi_1, \xi_2) N_{ij}(\xi_1, \xi_2) \frac{\partial^2 w^*}{\partial \xi_i \partial \xi_j} d\xi_1 d\xi_2 \tag{IV.125}$$

Now, for the rectangular plate, described above, which is simply supported on all four edges, $w^*(x, y, \xi_1, \xi_2)$ is given by (IV.73) with $\xi \to \xi_1, \eta \to \xi_2$. If we assume that the solution of the integral equation (IV.125) can be represented in series form as

$$w(x,y) = \sum_{i,k}^{\infty} A_{ik} \sin \alpha_i \sin \beta_k y, \tag{IV.126}$$

$\alpha_n = \dfrac{n\pi}{a}, \beta_m = \dfrac{m\pi}{b}$ (thus automatically satisfying the boundary conditions of simple support) then by substituting the series representations for w and w^* into (IV.125), and performing elementary computations, we are led to an infinite system of linear equations for the coefficients A_{ik} in (IV.126) of the form

$$\begin{gathered} A_{ik} + \frac{4}{abk(\alpha_i^2 + \beta_k^2)} \sum_{n,m} A_{mn} G_{nimk} = 0 \\ (i, k = 1, 2, \cdots, \infty) \end{gathered} \tag{IV.127}$$

where

$$G_{nimk} = \alpha_i^2 a_{nimk} + 2\alpha_i \beta_k c_{nimk} + \beta_k^2 b_{nimk}$$

with

$$\begin{cases} a_{nimk} &= \displaystyle\int_o^a \int_o^b N_{11}(\xi_1, \xi_2) \sin \alpha_i \xi_1 \sin \alpha_n \xi_1 \sin \beta_k \xi_2 \sin \beta_m \xi_2 d\xi_1 d\xi_2 \\ b_{nimk} &= \displaystyle\int_o^a \int_o^b N_{22}(\xi_1, \xi_2)) \sin \alpha_i \xi_1 \sin \alpha_n \xi_1 \sin \beta_k \xi_2 \sin \beta_m \xi_2 d\xi_1 d\xi_2 \\ c_{nimk} &= \displaystyle\int_o^a \int_o^b N_{12}(\xi_1, \xi_2) \cos \alpha_k \xi_1 \sin \alpha_n \xi_1 \cos \beta_k \xi_2 \sin \beta_m \xi_2 d\xi_1 d\xi_2 \end{cases}$$

The condition for buckling of the plate is, of course, that the (infinite) determinant of the system of equations (IV.117) be zero. A related (classical) treatment of the thermal buckling of a simply supported, isotropic, rectangular plate which employs the Rayleigh-Ritz procedure may be found in Klosner and Forray [13]; in [13] the temperature distribution is of the form (IV.119) and, in fact, is assumed to be symmetrical about the centerlines of the plate so that it is representable in the form of a double Fourier series in the functions $\cos \dfrac{n\pi x}{a} \cos \dfrac{m\pi y}{b}$ (the plate in [13] has length $2a$ and width $2b$).

There are many excellent treatments of the problem of general thermal deflections of an isotropic elastic, circular plate one of which may be found in [14]. We assume that the plate has radius $b > 0$ and is subjected to the radially symmetric subcase of (III.24), namely,

$$\delta T = T_o(r) + zT_1(r) \tag{IV.128}$$

It is also assumed in [14] that the edge of the plate is subjected to a uniform force P per unit length of the arc parameter s along the edge; for problems involving deflections due to temperature variations only we may set $P = 0$ in the results which follow below. Within the scope of small deflection theory it is easily seen that (IV.128) and (III.20) combine with the second equation in (III.23) so as to yield

$$\triangle^2 \Phi = -\alpha Eh \triangle T_o \tag{IV.129}$$

Clearly, if Φ_p is a solution of

$$\triangle \Phi_p = -\alpha Eh T_o \tag{IV.130}$$

it is also a solution of (IV.129. A particular solution of (IV.129 is, thus, obtained by using potential theory to integrate the Poisson's equation (IV.130), where $\triangle = \frac{1}{r}\frac{d}{dr}(r\frac{d}{dr})$ in view of the radial symmetry of T_o; we obtain

$$\Phi_p(r) = -\alpha Eh \int_o^r \left(\int_o^\xi T_o(\lambda)\lambda d\lambda\right)\frac{1}{\xi}d\xi \tag{IV.131}$$

The general solution of (IV.129) which satisfies the edge conditions delineated above may then be shown [14] to have the form

$$\begin{aligned} \Phi(r) = \frac{1}{2}\{-P + \frac{\alpha Eh}{r^2}\int_o^r T_o(\lambda)\lambda d\lambda\}r^2 \\ -\alpha Eh \int_o^r \left(\int_o^\xi T_o(\lambda)\lambda d\lambda\right)\frac{1}{\xi}d\xi \end{aligned} \tag{IV.132}$$

The first equation in the set (III.23) may now be written in the following form when $t \equiv 0$:

$$\begin{aligned} \triangle^2 w = -\Gamma \triangle T_1 + \frac{1}{K}\{N_r w_{,rr} + 2N_{r\theta}(\frac{1}{r}w_{,\theta})_{,r} \\ +N_\theta(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta})\} \end{aligned} \tag{IV.133}$$

where $N_r, N_\theta, N_{r\theta}$ are given in terms of Φ by (I.49) and

$$\Gamma = E\alpha h^3/12K(1-\nu)$$

In fact, with Φ as given by (IV.132) we have

$$\begin{cases} \sigma^o_{rr} &= -P + \dfrac{\alpha E}{b^2}\displaystyle\int_o^b T_o(\lambda)\lambda d\lambda - \dfrac{\alpha E}{r^2}\int_o^r T_o(\lambda)\lambda d\lambda \\ \sigma^o_{\theta\theta} &= -P + \dfrac{\alpha E}{b^2}\displaystyle\int_o^b T_o(\lambda)\lambda d\lambda + \dfrac{\alpha E}{r^2}\int_o^r T_o(\lambda)\lambda d\lambda \\ & \qquad -\alpha E T_o(r) \\ \sigma^o_{r\theta} &= 0 \end{cases} \tag{IV.134}$$

the superscripted o denoting, of course, that we are computing the middle surface stress field distribution. Because $N_r, N_\theta, N_{r\theta}$, and T_1 are independent of θ, it is easily seen that (IV.133) for the transverse deflection $w = w(r)$ may be rewritten in the form:

$$\begin{aligned} &\frac{d}{dr}(r\frac{d}{dr}[\frac{1}{r}\frac{d}{dr}\{r\frac{dw}{dr}\}]) \\ &\qquad = -\Gamma\frac{d}{dr}(r\frac{dT_1}{dr}) \\ &+\frac{h}{K}\Big[\left(P_0 r - \frac{\alpha E}{r}\int_0^r T_0(\lambda)\lambda d\lambda\right)\frac{d^2w}{dr^2} \\ &\qquad + \left(P_0 + \frac{\alpha E}{r^2}\int_0^r T_0(\lambda)\lambda d\lambda - \alpha E T_0\right)\frac{dw}{dr}\Big] \end{aligned} \tag{IV.135}$$

with the constant P_0 given by

$$P_0 = \frac{\alpha E}{b^2}\int_0^b T_0(\lambda)\lambda d\lambda - P \tag{IV.136}$$

An integration of (IV.135) now yields

$$\begin{aligned} \frac{d}{dr}\left\{\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right)\right\} &= -\Gamma\frac{dT_1}{dr} \\ &+\frac{h}{K}\left(P_0 - \frac{\alpha E}{r^2}\int_0^r T_0(\lambda)\lambda d\lambda\right)\frac{dw}{dr} + \frac{k_1}{r} \end{aligned} \tag{IV.137}$$

with k_1 a constant of integration. Thus, our problem has been reduced to that of solving the third order ordinary differential equation (IV.137) with appropriate boundary conditions. It is noted in [14] that further integrations of (IV.137) are not possible unless specific forms for $T_0(r)$ and $T_1(r)$ are assumed; in [14] these are taken as the truncated power series expansions

$$\begin{cases} T_0(r) = \dfrac{K}{\alpha E h} \sum_{j=0}^{n} t_{0j} r^j \\ T_1(r) = -\dfrac{1}{\Gamma} \sum_{j=0}^{n} t_{1j} r^j \end{cases} \tag{IV.138}$$

with the t_{0j}, t_{1j} real constants and m, n arbitrary positive integers. The solution of (IV.137) is now sought in the form of a power series. Substituting (IV.138) in (IV.137) and setting

$$c_j = \frac{t_{0j}}{j+2}, \; d_j = j t_{1j}$$

we obtain

$$\begin{aligned} \frac{d^3 w}{dr^3} + \frac{d^2 w}{dr^2} + \Big\{ \sum_{j=0}^{m} c_j (r^j - b^j) \\ + \frac{hP}{K} - \frac{1}{r^2} \Big\} \frac{dw}{dr} = \sum_{j=1}^{n} d_j r^{j-1} \end{aligned} \tag{IV.139}$$

Setting

$$\begin{cases} u(r) = \dfrac{dw}{dr} \\ b_0' = \dfrac{hP}{K} - \sum_{j=1}^{m} c_j b^j \end{cases}$$

now yields the equation

$$\frac{d^2 u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left(\sum_{j=1}^{m} c_j r^j + b_0' - \frac{1}{r^2} \right) u = \sum_{j=0}^{n-1} d_{j+1} r^j \tag{IV.140}$$

for $u = u(r)$. Relative to (IV.140) we now consider solutions of the form

$$u(r) = r \sum_{i=0}^{\infty} \lambda_i r^i \tag{IV.141}$$

Inserting (IV.141) in (IV.140) a recurrence relation of the form

$$(j+4)(j+2)\lambda_{j+2} + b_0'\lambda_j + \sum_{i=0}^{m} c_j\lambda_{j-1} = \begin{cases} d_{j+2},\ j=-1,0,...,n-2 \\ 0,\ j=n-1,n,... \end{cases} \tag{IV.142}$$

is generated for the λ_j with $\lambda_j = 0$ if $j < 0$. A careful discussion of the convergence of the series (IV.141), with the λ_j as given by (IV.142), may be found in [14]. Inserting (IV.141) into the equation $\frac{dw}{dr} = u(r)$, and integrating, we find that the transverse deflection may be represented in the form

$$w(r) = r^2 \sum_{i=0}^{\infty} \kappa_i r^i + \kappa \tag{IV.143}$$

with κ a constant of integration and $\kappa_i = \lambda_i/(i+2)$. By referring to (IV.142) it may be deduced that

$$\kappa_i = \lambda_0 \xi_i + \delta_i$$

where ξ_i contains the parameters P, K, h, b and some (or all) of the t_{0j} while δ_i contains these parameters as well as some (or all) of the t_{1j}. Thus (IV.143) may be rewritten in the form

$$w(r) = \lambda_0 w_0(r) + w_1(r) + r \tag{IV.144}$$

where

$$w_0(r) = \sum_{j=0}^{\infty} \xi_j r^j,\ w_1(r) = \sum_{j=0}^{\infty} \delta_j r^j \tag{IV.145}$$

It is easily seen that $w_1(r)$ is a particular solution of (IV.139) while $w_0(r)$ is the solution of the corresponding homogeneous equation which is bounded at $r = 0$.

The constants λ_0 and r in (IV.144) are determined by the support conditions along the edge of the plate at $r = b$; these support conditions for the clamped edge and the simply supported edge are, respectively, in view of (III.26)

$$w'(b) = 0,\ w(b) = 0 \tag{IV.146a}$$

and

$$\begin{cases} K[w''(b) + \dfrac{\nu}{b}w'(b)] + \Gamma T_1(b) = 0 \\ w(b) = 0 \end{cases} \tag{IV.146b}$$

We will proceed by considering the clamped edge conditions only; the analysis corresponding to the simply supported conditions in [14] would appear to be correct only if T_1 vanishes along the edge of the plate at $r = b$.

Substituting (IV.144) into (IV.146a) yields, for the clamped plate

$$\lambda_0 = -\frac{w_1'(b)}{w_0'(b)}, \quad r = \frac{w_1'(b)}{w_0'(b)}w_0(b) - w_1(b) \tag{IV.147}$$

Inserting the values of λ_0, κ in (IV.137) into (IV.144) we obtain for the plate which is clamped along the edge at $r = b$

$$w(r) = \frac{w_1'(b)}{w_0'(b)}\{w_0(b) - w_0(r)\} + w_1(r) - w_1(b) \tag{IV.148}$$

It will be assumed that $w_0'(b) \neq 0$; as noted in [14] if $w_0'(b) = 0$ then b is the radius of the clamped plate with given temperature distribution $t_0(r)$ for which P is a critical buckling load.

In the special case in which $m = 0$, $T_0(r)$ reduces, in light of (IV.138), to a constant while (IV.140) becomes

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \left(\frac{hP}{K} - \frac{1}{r^2}\right)u = \sum_{j=0}^{n-1} d_{j+1}r^j \tag{IV.149}$$

Using (IV.141) again, and working with the special case of the recurrence relation (IV.142) for $m = 0$, it is not difficult to show that the transverse deflection assumes the form

$$w(r) = \lambda_0 J_0\left(r\sqrt{\frac{hP}{K}}\right) + w_{10}(r) + \kappa \tag{IV.150}$$

with J_0 the Bessel function of the first kind of order zero and $w_{10}(r)$ the form assumed by $w_1(r)$ for $m = 0$. Applying the boundary conditions (IV.146a) we now find as the expression for the transverse deflection of the circular, isotropic plate clamped along its edge at $r = b$

$$w(r) = \frac{w_{10}'(b)}{\sqrt{\frac{h}{K}} J_0\left(b\sqrt{\frac{hP}{K}}\right)} \left\{ J_0\left(r\sqrt{\frac{hP}{K}}\right) - J_0\left(b\sqrt{\frac{hP}{K}}\right) \right\} + w_{10}(r) - w_{10}(b) \tag{IV.151}$$

The analysis of the isotropic circular plate presented above did not involve any considerations of plate stability; as has already been indicated the thermal buckling problem within the context of small deflection theory is mathematically equivalent to the initial (thermal) buckling problem without the small deflection assumption. For an isotropic plate the thermal stress resultant N^T, as given by (I.11a) may be written in the form

$$N^T = \frac{\alpha E h}{1-\nu} T_m(x, y) \tag{IV.152}$$

where

$$T_m(x, y) = \frac{1}{h} \int_{-h/2}^{h/2} \delta T(x, y, z)\, dz \tag{IV.153}$$

may be thought of as the medium temperature in the plate. Suppose that

$$\int_{-h/2}^{h/2} \delta T(x, y, z) z\, dz = 0$$

so that $M^T = 0$. The buckling problem (in rectangular coordinates), either assuming small deflection theory or focusing on the initial buckling problem, then takes the form (see (III.4a,b))

$$\begin{aligned} \Delta^2 \Phi &= -\alpha E h \Delta T_m \\ \Delta^2 w &= \frac{1}{K}(\Phi_{,yy} w_{,xx} - 2\Phi_{,xy} w_{,xy} + \Phi_{,xx} w_{,yy}) \end{aligned} \tag{IV.154}$$

if we, again, assume that $t \equiv 0$. Following the stability analysis in [15] we let T_0 represent the maximum value (upper bound) of T_m on the domain of the plate so that $0 \leq T_m \leq T_0$ and we introduce the dimensionless temperature parameter $\mathcal{T} = T_m/T_0$ so that $0 \leq \mathcal{T} \leq 1$. If b represents a characteristic length associated with the plate then, in rectangular coordinates, we may introduce the dimensionless variables $\xi = x/b$, $\eta = y/b$ in which case (IV.154) would assume the form

$$\begin{aligned} \Delta^2\phi &= -\Delta\mathcal{T} \\ \Delta^2 w &= \lambda(\phi_{,\eta\eta}w_{,\xi\xi} - 2\phi_{,\xi\eta}w_{,\xi\eta} + \phi_{,\xi\xi}w_{,\eta\eta}) \end{aligned} \tag{IV.155}$$

where

$$\phi = \Phi/b^2\alpha EhT_0 \tag{IV.156}$$

while, by virtue of the definition of K,

$$\lambda = 12(1-\nu^2)\left(\frac{b}{h}\right)^2 \alpha T_0 \tag{IV.157}$$

and, of course, $\Delta = \dfrac{\partial^2}{\partial\xi^2} + \dfrac{\partial^2}{\partial\eta^2}$. Following the discussion in [15], the edge(s) of the plate are assumed to be free of applied forces and moments so that the buckling (stability) problem consists of finding the temperature parameter $\mathcal{T}$ which minimizes the value of λ in (IV.155); this may be achieved, e.g., by following the Raleigh-Ritz procedure and considering the expression

$$\Lambda = \Lambda_N/\Lambda_D \tag{IV.158}$$

with

$$\begin{cases} \Lambda_N = \displaystyle\iint_{\mathcal{A}} \left\{(w_{,\xi\xi} + w_{,\eta\eta})^2 - 2(1-\nu)[w_{,\xi\xi}w_{,\eta\eta} - w_{,\xi\eta}^2]\right\} d\xi d\eta \\ \Lambda_D = \displaystyle\iint_{\mathcal{A}} \left\{2\phi_{,\xi\eta}w_{,\xi}w_{,\eta} - \phi_{,\eta\eta}w_{,\xi}^2 - \phi_{,\xi\xi}w_{,\eta}^2\right\} d\xi d\eta \end{cases}$$

where $\mathcal{A}$ is the domain occupied by the midplane of the plate. As $\lambda \leq \Lambda$ for all admissible functions $w = w^*(\xi,\eta)$, which satisfy the edge support conditions, and for any such w^*, Λ_N remains constant while Λ_D is a functional of ϕ the buckling problem may be cast in the following form: compute the

$$\max_{w^*} \iint_{\mathcal{A}} \left\{2\phi_{,\xi\eta}w^*_{,\xi}w^*_{,\eta} - \phi_{,\eta\eta}w^{*2}_{,\xi} - \phi_{,\xi\xi}w^{*2}_{,\eta} - \phi_{,\xi\xi}w^{*2}_{,\eta}\right\} d\xi d\eta$$

for $\phi(\xi,\eta)$ satisfying (IV.155), subject to $0 \leq \mathcal{T} \leq 1$.

We now return to the problem of thermal buckling of an isotropic, elastic circular plate and apply the general methodology elucidated above to the special case of a circular plate subjected to a radially symmetric (nondimensional) temperature field $\mathcal{T}$; as in [15], however, the buckling mode will be allowed to depend on the angular coordinate θ. As the Airy function Φ is also independent of θ the membrane equation (IV.154) reduces to an ordinary differential equation in the radial coordinate r; setting $G = F'(r)$ this equation becomes

$$G''(r) + \frac{1}{r}G'(r) - \frac{1}{r^2}G = -E\alpha h \frac{dT_m}{dr} \tag{IV.159}$$

and

$$N_r = \frac{1}{r}G, \; N_\theta = G'(r) \tag{IV.160}$$

Using the plate radius b as a reference length and introducing the nondimensional coordinate $\rho = r/b$ we may rewrite (IV.159) in the form

$$g''(\rho) + \frac{1}{\rho}g'(\rho) - \frac{1}{\rho^2}g = -\frac{d\mathcal{T}}{d\rho} \tag{IV.161}$$

with

$$g(\rho) = G(\rho)/E\alpha h b T_0 \tag{IV.162}$$

The solution of (IV.161) which satisfies the boundary condition $N_r(b) = 0$, and also satisfies the condition that $N_r(0)$ and $N_\theta(0)$ remain bounded, is

$$g(\rho) = -\frac{1}{\rho}\int_0^\rho \mathcal{T}(\lambda)\lambda d\lambda + \rho \int_0^1 \mathcal{T}(\lambda)\lambda d\lambda \tag{IV.163}$$

As noted in [15], it may be demonstrated that the solution defined by (IV.163) remains valid for a noncontinuous temperature distribution. For the problem at hand, (IV.158) can be written in the following form

$$\Lambda = \frac{\int_0^l \int_0^{2\pi} [(\Delta w)^2 - (1-\nu)L(w,w)]\rho d\rho d\theta}{-\int_0^l \int_0^{2\pi} \left[\frac{1}{\rho}g(\rho)\left(\frac{\partial w}{\partial \rho}\right)^2 + \frac{1}{\rho^2}\frac{dg}{d\rho}\left(\frac{\partial w}{\partial \theta}\right)^2\right]\rho d\rho d\theta} \tag{IV.164}$$

where

$$\Delta w = \frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial w}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \theta^2}$$

and

$$\begin{aligned} L(w,w) = {} & 2\left(\frac{1}{\rho}\frac{\partial w}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 w}{\partial \theta^2}\right)\frac{\partial^2 w}{\partial \rho^2} \\ & -2\left(\frac{1}{\rho}\frac{\partial^2 w}{\partial \rho \partial \theta} - \frac{1}{\rho^2}\frac{\partial w}{\partial \theta}\right)^2 \end{aligned} \tag{IV.165}$$

Following the analysis in [15] we now introduce the admissible functions

$$w = w^*(\rho,\theta) = w_m(\rho)\cos m\theta, \;\; m = 0, 1, 2... \tag{IV.166}$$

which are subject to the boundary conditions

$$m_r(b) = 0, \;\; \left[q_r + \frac{1}{b}\frac{\partial m_{r\theta}}{\partial \theta}\right]_{r=b} = 0 \tag{IV.167}$$

where q_r, m_r, and $m_{r\theta}$ are (in r, θ coordinates) the z-direction shear force, bending moment, and twisting moment, respectively, per unit length along the edge at $r = b$. With respect to the set of admissible functions in (IV.166), the boundary conditions (IV.167) become

$$\left[\frac{d^2 w_m}{d\rho^2} + \nu\left(\frac{dw_m}{d\rho} - m^2 w_m\right)\right]_{\rho=1} = 0 \tag{IV.168a}$$

$$\left\{\frac{d^3 w_m}{d\rho^3} + 2\frac{d^2 w_m}{d\rho^2} + [1 + m^2(2-\nu)]\frac{dw_m}{d\rho} + m^3(3-\nu)w_m\right\}_{\rho=1} = 0 \tag{IV.168b}$$

To the conditions (IV.167) we append

$$w(0) = 0, \left.\frac{dw}{dr}\right|_{r=0} = 0$$

or

$$\left.w_m\right|_{\rho=0} = 0, \;\; \left.\frac{dw_m}{d\rho}\right|_{\rho=0} = 0 \tag{IV.168c}$$

which has the effect of eliminating undetermined rigid body displacements of the plate.

By introducing the $w^*(\rho,\theta)$ in (IV.166) into (IV.164) we obtain for Λ the expression

$$\Lambda = \frac{\int_0^1 \left\{[\Delta_m(w_m)]^2 - (1-\nu)L_m(w_m,w_m)\right\}\rho d\rho}{-\int_0^1 \left[g(\rho)\left(\frac{dw_m}{d\rho}\right)^2 + \frac{m^2}{\rho}\frac{dg}{d\rho}w_m^2\right]d\rho} \tag{IV.169}$$

where

$$\Delta_m(w_m) = \frac{d^2 w_m}{d\rho^2} + \frac{1}{\rho}\frac{w_m}{d\rho} - \frac{m^2}{\rho^2}w_m \tag{IV.170a}$$

and

$$\begin{aligned} L_m(w_m,w_m) = 2\Bigg[&\left(\frac{1}{\rho}\frac{dw_m}{d\rho} - \frac{m^2}{\rho^2}w_m\right)\frac{d^2w_m}{d\rho^2} \\ &-m^2\left(\frac{1}{\rho}\frac{dw_m}{d\rho} - \frac{1}{\rho^2}w_m\right)^2\Bigg] \end{aligned} \tag{IV.170b}$$

Therefore, for any $w_m(\rho)$ satisfying (IV.168a,b,c) and for any given wave number m of the buckling mode, Λ is minimized by requiring that the denominator of (IV.169) be a maximum over all functions $g(\rho)$ of the form (IV.163) for $0 \le \mathcal{T} \le 1$. We recall that the circumferential membrane force $N_\theta(\rho)$ must, because of the planar equilibrium conditions, change its sign at least once in the interval $0 \le \rho \le 1$, i.e., ([7], §13), $\int_0^1 N_\theta(\rho)d\rho = 0$. If, as in [15], we assume that there exists but one change of sign of $N_\theta(\rho)$ at the value $\rho = \mu$ then using the relations (IV.160) as applied to g, in lieu of G, we have the following:

1) For a plate heated near its center at $\rho = 0$

$$\begin{cases} \frac{dg}{d\rho} \le 0, \; g \le 0, \; \text{for } 0 \le \rho \le \mu \\ \frac{dg}{d\rho} \ge 0, \; g \le 0, \; \text{for } \mu \le \rho \le 1 \end{cases} \tag{IV.171a}$$

2) For a plate heated near its edge at $\rho = 1$

$$\begin{cases} \frac{dg}{d\rho} \ge 0, \; g \ge 0, \; \text{for } 0 \le \rho \le \mu \\ \frac{dg}{d\rho} \le 0, \; g \ge 0, \; \text{for } \mu \le \rho \le 1 \end{cases} \tag{IV.171b}$$

From (IV.163) we compute

$$\frac{dg}{d\rho} = \frac{1}{\rho^2}\int_0^\rho \mathcal{T}(\lambda)\lambda d\lambda - \mathcal{T}(\rho) + \int_0^1 \mathcal{T}(\lambda)\lambda d\lambda \tag{IV.172}$$

so that, by virtue of (IV.171a,b)

$$\frac{1}{\mu^2}\int_0^\mu \mathcal{T}(\lambda)\lambda d\lambda - \mathcal{T}(\mu) + \int_0^1 \mathcal{T}(\lambda)\lambda d\lambda = 0 \tag{IV.173}$$

Equation (IV.173) serves to determine all points $\rho = \mu$ at which a change of sign of $N_\theta(\rho)$, i.e. $g'(\rho)$, may occur, it being understood that the assumption that $N_\theta(\rho)$ can change sign at only one point in the interval $0 \le \rho \le 1$ places a restriction on the temperature distribution.

We now write the denominator Λ_0 in (IV.169) in the form

$$\begin{aligned}\Lambda_0 = &\int_0^\mu \left[g(\rho)\left(\frac{dw_m}{d\rho}\right)^2 + \frac{m^2}{\rho}\frac{dg}{d\rho}w_m^2\right] d\rho \\ &- \int_\mu^1 \left[g(\rho)\left(\frac{dw_m}{d\rho}\right)^2 + \frac{m^2}{\rho}\frac{dg}{d\rho}w_m^2\right] d\rho\end{aligned}$$

and effect the same splitting in both (IV.163) and (IV.172), i.e., if we set

$$\begin{cases} I = \displaystyle\int_0^\mu \mathcal{T}(\lambda)\lambda d\lambda \\ II = \displaystyle\int_\mu^1 \mathcal{T}(\lambda)\lambda d\lambda \end{cases} \tag{IV.174}$$

then

$$\begin{cases} g = -\dfrac{1}{\rho}\displaystyle\int_0^\rho \mathcal{T}(\lambda)\lambda d\lambda + \rho I + \rho II \\ \dfrac{dg}{d\rho} = \dfrac{1}{\rho^2}\displaystyle\int_0^\rho \mathcal{T}(\lambda)\lambda d\lambda - \mathcal{T}(\rho) + I + II \end{cases} , \quad 0 \le \rho \le \mu \tag{IV.175}$$

and

$$\begin{cases} g = -\dfrac{1}{\rho}\displaystyle\int_\mu^\rho \mathcal{T}(\lambda)\lambda d\lambda + \rho II - \left(\dfrac{1}{\rho} - \rho\right) I \\ \dfrac{dg}{d\rho} = \dfrac{1}{\rho^2}\displaystyle\int_\mu^\rho \mathcal{T}(\lambda)\lambda d\lambda - \mathcal{T}(\rho) + \left(\dfrac{1}{\rho^2} + 1\right) I + II \end{cases} , \quad \mu \le \rho \le 1 \tag{IV.176}$$

If we begin by first considering a radially symmetric buckling mode then, by (IV.166), $m = 0$ and Λ_D reduces to

$$\Lambda_D\Big|_{m=0} = -\int_0^\mu g(\rho)\left(\frac{dw_0}{d\rho}\right)^2 d\rho - \int_\mu^1 g(\rho) - \int_\mu^1 g(\rho)\left(\frac{dw_0}{d\rho}\right)^2 d\rho \tag{IV.177}$$

where, for a plate heated near its center at $\rho = 0$, $g \le 0$. As in [15], we assume that the temperature field $\mathcal{T}$ is restricted so as to satisfy

$$-\frac{1}{\rho}\int_{\mu}^{\rho} \mathcal{T}(\lambda)\lambda d\lambda + \rho II \ge 0 \tag{IV.178}$$

which prohibits the presence of strong oscillation in $\mathcal{T}$. Using (IV.178) it follows from the form of $g(\rho)$ in (IV.175), (IV.176) that $-g(\rho)$ is maximized for $0 \le \rho \le 1$ by setting $\mathcal{T}(\rho) \equiv 0$ for $\mu \le \rho \le 1$; this latter condition now reduces $g(\rho)$ to the form

$$g(\rho) = \begin{cases} -\dfrac{1}{\rho}\displaystyle\int_0^{\rho} \mathcal{T}(\lambda)\lambda d\lambda + \rho I, \ 0 \le \rho \le \mu \\ -\left(\dfrac{1}{\rho} - \rho\right) I, \ \mu \le \rho \le 1 \end{cases} \tag{IV.179}$$

However, for $\mu \le \rho \le 1$, it is clear that $-g(\rho)$ is maximized by maximizing I and, in view of (IV.174), we obtain (set $\mathcal{T}(\lambda) \equiv 1$, $0 \le \lambda \le \mu$)

$$g(\rho) = -\left(\frac{1}{\rho} - \rho\right)\frac{\mu^2}{2}, \ \mu \le \rho \le 1 \tag{IV.180}$$

An analogous result holds for $-g(\rho)$ with ρ in the interval $[0, \mu]$, namely, we have

$$g(\rho) = -\frac{\rho}{2}(1 - \mu^2), \ 0 \le \rho \le \mu \tag{IV.181}$$

By introducing (IV.180) and (IV.181) into (IV.177) we see that the function to be maximized becomes

$$\Lambda_D\Big|_{m=0} = \frac{1}{2}(1 - \mu^2)\int_0^{\mu} \rho \left(\frac{dw_m}{d\rho}\right)^2 d\rho + \frac{1}{2}\mu^2 \int_{\mu}^{1} \left(\frac{1}{\rho} - \rho\right)\left(\frac{dw_m}{d\rho}\right)^2 d\rho \tag{IV.182}$$

As $0 \le \rho \le 1$, and $0 < \mu < 1$, the expression on the right-hand side of (IV.182) is positive; for any admissible function $\Lambda_D\Big|_{m=0}$ may be computed and, moreover, the value of μ can be determined which maximizes $\Lambda_D\Big|_{m=0}$.

If we retain the same temperature field, i.e., $\mathcal{T} = 1$ for $0 \le \rho \le \mu$ and $\mathcal{T} \equiv 0$ for $\mu \le \rho \le 1$ but consider buckling modes which are not radially symmetric ($m \neq 0$) the situation described above changes. In this case, it follows from (IV.175), (IV.176) that

$$\frac{dg}{d\rho} = \begin{cases} -\frac{1}{2}(1-\mu^2), \; 0 \le \rho \le \mu \\ \frac{1}{2}\mu^2\left(\frac{1}{\rho^2}+1\right), \; \mu \le \rho \le 1 \end{cases} \tag{IV.183}$$

so that, as per (IV.171a), $\frac{dg}{d\rho} \ge 0$, for $\mu \le \rho \le 1$ while $\frac{dg}{d\rho} \le 0$, for $0 \le \rho \le \mu$. Using (IV.180), (IV.181), and (IV.183), it follows that the function to be maximized is given by

$$\begin{aligned} \Lambda_D = & \frac{1}{2}(1-\mu^2)\left[\int_0^\mu \rho\left(\frac{dw_m}{d\rho}\right)^2 d\rho + m^2\int_0^\mu \frac{w_m^2}{\rho}d\rho\right] \\ & +\frac{1}{2}\mu^2\Big[\int_\mu^1 \left(\frac{1}{\rho}-\rho\right)\left(\frac{dw_m}{d\rho}\right)^2 d\rho \\ & -m^2\int_\mu^1\left(\frac{1}{\rho^2}+1\right)\frac{w_m^2}{\rho}d\rho\Big] \end{aligned} \tag{IV.184}$$

whic, for arbitrary m, must be a positive quantity. However, if we consider admissible functions $w_m(\rho)$ of the (polynomial) form

$$w_m(\rho) = C\rho^2(1+c_1\rho+c_2\rho^2) \tag{IV.185}$$

satisfying all the boundary conditions in (IV.168a,b,c,d) then the right-hand side of (IV.184) will be negative for $m \ge 2$ and, thus, no buckling mode with $m \ge 2$ is possible if $\mathcal{T} \equiv 1$, for $0 \le \rho \le \mu$, while $\mathcal{T} \equiv 0$, for $\mu \le \rho \le 1$; rather, buckling modes corresponding to wave numbers $m \ge 2$ are caused by circumferential compressive stresses in a neighborhood of the edge of the plate at $r = b$ (i.e., the relevant conditions with respect to $g(\rho)$ are those in (IV.171b)). It is, in fact, shown in [15] that corresponding to the temperature field $\mathcal{T}$ (within the domain of piecewise constant temperature fields) $-\frac{dg}{d\rho}$ is maximized by the field $\mathcal{T} \equiv 0$, for $0 \le \rho \le \mu$, $\mathcal{T} \equiv 1$, for $\mu \le \rho \le 1$, and

$$g(\rho) = \begin{cases} \frac{1}{2}\rho(1-\mu^2), \; 0 \le \rho \le \mu \\ \left(\frac{1}{\rho}-\rho\right)\frac{\mu^2}{2}, \; \mu \le \rho \le 1 \end{cases} \tag{IV.186a}$$

and

$$\frac{dg}{d\rho} = \begin{cases} \frac{1}{2}(1-\mu^2), \ 0 \le \rho \le \mu \\ -\left(\frac{1}{\mu^2}+1\right)\frac{\mu^2}{2}, \ \mu \le \rho \le 1 \end{cases}$$

In lieu of (IV.184) we now obtain

$$\begin{aligned} \Lambda_D = -\frac{1}{2}(1-\mu^2)&\left[\int_0^\mu \rho\left(\frac{dw_m}{d\rho}\right)^2 d\rho + m^2\int_0^\mu \frac{w_m^2}{\rho}d\rho\right] \\ +\frac{1}{2}\mu^2&\left[-\int_\mu^1\left(\frac{1}{\rho}-\rho\right)\left(\frac{dw_m}{d\rho}\right)^2 d\rho + m^2\int_\mu^1\left(\frac{1}{\rho^2}+1\right)\frac{w_m^2}{\rho}d\rho\right] \end{aligned} \qquad \text{(IV.187)}$$

as the function to be maximized. The expression in (IV.187) can now be used, in conjunction with a class of admissible $w_m(\rho)$, to determine μ for $m \geq 2$; if we use that class of admissible functions $w_m(\rho)$ which is defined by (IV.185), we obtain the results shown in Table 1. A related treatment of the thermal buckling of isotropic, circular elastic plates may be found in [10] and further discussion of several aspects of initial buckling will be presented in the next section. An excellent discussion of closed-form representations of the stability boundary associated with two-dimensional temperature fields (in simply supported elastic rectangular plates) that produce combined compression, tension and shear as well as with temperature fields without shear, and one-dimensional temperature fields, may be found in Bargmann [16]. We will return to the problem of initial buckling of a thermally loaded elastic circular plate in §V.

V. THERMAL BENDING AND BUCKLING OF AND POST-BUCKLING BEHAVIOR - LARGE DEFLECTION THEORY

In this section we will relinquish the small deflection assumption that was imposed in §IV; we will work within the context of large deflection theory and will study the buckling and postbuckling behavior of isotropic and orthotropic elastic plates. For the case of a rectangular plate our focus will be on the postbuckling behavior in the isotropic case and on large thermal deflections in the orthotropic case; postbuckling problems for rectangular orthotropic plates will be considered in Section VI within the context of Berger's approximation. For the case of an isotropic circular plate the emphasis in this section will be on computing the critical buckling temperature and the corresponding buckling mode; a discussion of initial buckling, within the context of thermoelasticity theory, for cylindrically orthotropic circular plates may be found in the paper by Stavsky [17].

The solution of a large deflection problem for an isotropic rectangular plate involves the determination of the two unknown functions $w(x, y)$ and $\Phi(x, y)$ from equations (III.4a,b) and suitable boundary support conditions. In the small deflection case the buckling problem could be solved in two steps, namely, one would solve (III.4b), with $[w, w] = 0$, for Φ and then employ that solution in (III.4a) to independently determine w. In the present situation a simultaneous solution for both Φ and w is needed and in most situations one must resort to (approximate) series solution, to iteration schemes, or to numerical methods. An iterative procedure due to S.R. Boley [18], which has been described in [7], will now be presented for the case of thermal buckling of a rectangular isotropic plate; in fact, the temperature distribution and applied boundary conditions on the edges are identical with those in (IV.93) - (IV.95), respectively, so that for edges at $x = 0, a (0 \leq y \leq b)$ and $y = 0, b (0 \leq x \leq a)$ simply supported, i.e. (IV.101), the critical combination of applied edge stress σ_0 and 'temperature' c_1 is obtained by setting the determinant of the system of homogeneous linear algebraic equations (IV.103) equal to zero, where k_{mn} is given by (IV.104).

The iterative procedure consists of determining a set of successive approximations where the first approximation is the linear solution, i.e., those expressions for Φ and w which correspond to the loading that initiates buckling; these expressions are then employed in the nonlinear terms of (III.4a,b) in the manner described below, with $t \equiv 0$, to obtain new expressions for Φ and w and successive iteratives are computed in the same fashion. Provided that the actual solution is close to the initial iterative (i.e., the loading does not greatly exceed the critical load) the convergence of the iterative scheme may be shown to be rapid [18]; in fact, as little as two iterations may suffice to produce reasonable results. Suppose that we retain only two terms of the infinite series for w, i.e., of (IV.102); then, as shown in [7] the deflected shape, at this level of approximation, assumes the form

$$w = \sin\frac{m\pi x}{a}\left\{a_1\frac{\sin m\pi y}{a} + a_3\frac{\sin 3m\pi y}{a}\right\} \tag{V.1}$$

As a first approximation we take $\left(\frac{a_3}{a_1}\right) = L_3$ with L_3 the solution of the equation which corresponds to either the first or second line of (IV.105); thus

$$L_3 = 1 - 2\left(\frac{k_{m1} + \sigma_0}{\alpha E c_1}\right) \tag{V.2}$$

The first approximation for w in the iterative scheme say, $w^{(1)}$, is then given by (V.1), with a_3/a_1 given by (V.2); it is, therefore, expressed in terms of an amplitude a_1 which is still indeterminate at this point. To obtain the second iteration, we substitute the first set of iteratives, i.e $w^{(1)}(x,y)$, as determined above, and $\Phi^{(1)}(x,y)$, as given by (IV.97) into the system (III.4a,b) as follows; first $w^{(1)}$ is used in (III.4b) so as to produce the equation

$$\triangle^2\,\Phi^{(2)} = -\frac{1}{2}Eh[w^{(1)}, w^{(1)}] - (1-\nu)N^T \tag{V.3}$$

for $\Phi^{(2)}(x,y)$; the solution of (V.3) which satisfies the same boundary conditions as $\Phi^{(1)}(x,y)$ is given by

$$\begin{aligned}\Phi^{(2)} &= \frac{\sigma_0 y^2}{2} - \sum_{n=2,4,6} r_n \cos\left(\frac{nm\pi y}{a}\right)\\ &\quad + \cos\left(\frac{2m\pi x}{a}\right)\sum_{n=0,2,4} s_n \cos\left(\frac{nm\pi y}{a}\right)\end{aligned} \tag{V.4}$$

with

$$\begin{cases} r_2 = \dfrac{-\alpha E c_1 a^2}{2m^2\pi^2} + \dfrac{Ea_1^2}{32}(1-2L_3) \\ r_4 = \dfrac{Ea_1^2}{64}L_3 \\ r_6 = \dfrac{Ea_1^2}{288}L_3^2 \end{cases} \tag{V.5a}$$

and

$$\begin{cases} s_0 = \dfrac{Ea_1^2}{32}(1+9L_3^2) \\ s_2 = \dfrac{Ea_1^2}{16}L_3 \\ s_4 = -\dfrac{Ea_1^2}{400}L_3 \end{cases} \tag{V.5b}$$

Next $w^{(1)}$ and $\Phi^{(2)}$ are substituted into the right-hand side of (III.4a) so as to produce for $w^{(2)}(x, y)$ the equation

$$\begin{aligned} K \, \triangle^2 \, w^{(2)} &= \Phi^{(2)}_{,yy} w^{(1)}_{,xx} - 2\Phi^{(2)}_{,xy} w^{(1)}_{,xy} \\ &+\Phi^{(2)}_{,xx} w^{(1)}_{,yy} - \triangle M^T \end{aligned} \tag{V.6}$$

Solving (V.6), subject to the support conditions, we now obtain $w^{(2)}$ as an infinite series involving the constants a_1 and a_3, which are now subject to the conditions

$$\begin{aligned} &\left(k_{m1} + \sigma_0 - \frac{\alpha E c_1}{2}\right) + \frac{\alpha E c_1}{2}\left(\frac{a_3}{a_1}\right) \\ &\quad = 3(1-\nu^2)k_{m1}\left(\frac{a_1}{h}\right)^2\left(2 - 3L_3 + \frac{3801}{400}L_3^2\right) \end{aligned} \tag{V.7a}$$

and

$$\begin{aligned} &\left(\frac{\alpha E c_1}{2}\right) + (k_{m3} + \sigma_0)\left(\frac{a_3}{a_1}\right) \\ &\quad = 3(1-\nu^2)k_{m1}\left(\frac{a_1}{h}\right)^2\left(-1 + \frac{426}{25}L_3\right) \end{aligned} \tag{V.7b}$$

By solving the (simultaneous) system (V.7a,b) for a_1, a_3 we obtain $w^{(2)}$ thus producing, for the problem at hand, the second iteration $(w^{(2)}, \Phi^{(2)})$ which is based on retaining just two terms in the series for $w(x, y)$. In Fig. 10 we show some of the corresponding numerical results for the variation of the deflection at the center point of the panel with increasing thermal load. It is indicated in [7] that if one applies the scheme described above, but retains three terms in the series for the deflection $w(x, y)$, and proceeds through the third iteration $(w^{(3)}, \Phi^{(3)})$, then for the range of thermal loading displayed in Fig. 10 the results are practically identical to those determined above.

One of the earliest and most frequently referenced studies of thermal buckling and postbuckling of rectangular isotropic plates is the NACA Technical Note [2] by Gossard, Seide, and Roberts which treats buckling of a simply supported plate that is subjected to a tentlike temperature distribution; this paper also deals with the effects of initial imperfections on the buckling analysis, a subject that we will cover, briefly, in Chapter VII. The rectangular plate in [2] is heated along its' longitudinal center line by a uniform line source of heat and cooled along its' edges by two uniform, equal, line sinks of heat; such an arrangement yields a temperature distribution in the plate which is constant through the thickness and which varies in a tentlike manner over the face of the plate (Figs. 11a, 11b). All edges of the plate are restrained in a direction normal to the plane of the plate by simple rigid supports but are free to move in the plane of the plate.

The analysis in [2] proceeds by first computing the thermal stresses at temperature levels below the critical level after which the critical (buckling) temperature is determined; then the (postbuckling) behavior of the plate at temperature levels above the critical level is analyzed. The actual details of the calculation of the thermal stress distribution in the plate of reference [2] are given in [19] and are obtained by employing the first order approximation that on any cross section normal to the x-axis the stress component σ_{xx} is distributed as in Fig. 12. The (prebuckling) Airy function $\tilde{\Phi}_0$ based on the stress distribution (instead of on the resultant or averaged stresses) may, in this case, be expressed as the product of a known function of y and an arbitrary function of x; its approximate form, for $0 \leq y \leq b$, is

$$\begin{aligned} \tilde{\Phi}_0(x,y) = \frac{1}{12} b^2 E\alpha T_0 \Big(1 - 3\frac{y^2}{b^2} + 2\frac{y^3}{b^3}\Big) \\ x\Big(B_1 \sin hR_1 \frac{x}{a} \sin R_2 \frac{x}{a} + \\ B_2 \cos hR_1 \frac{x}{a} \cos R_2 \frac{x}{a} + 1\Big) \end{aligned} \tag{V.8}$$

with the constants B_1, B_2, R_1, R_2 as defined in appendix A of [2], i.e.,

$$R_1 = k_1 \frac{a}{b}, \quad R_2 = k_2 \frac{a}{b} \tag{V.9a}$$

$$\begin{cases} k_1 = \sqrt[4]{\dfrac{105}{13}} \cdot \sqrt{1 + \sqrt{\dfrac{21}{65}}} \\ \\ k_2 = \sqrt[4]{\dfrac{105}{13}} \cdot \sqrt{1 - \sqrt{\dfrac{21}{65}}} \end{cases} \tag{V.9b}$$

$$\begin{cases} B_1 = \dfrac{k_1 \sin hR_1 \cos R_2 - k_2 \cos hR_1 \sin R_2}{k_1 \sin R_2 \cos R_2 + k_2 \sin hR_1 \cos hR_2} \\ B_2 = -\dfrac{k_1 \cos hR_1 \sin R_2 - k_2 \sin hR_1 \cos R_2}{k_1 \sin R_2 \cos R_2 + k_2 \sin hR_1 \cos hR_2} \end{cases} \tag{V.9c}$$

Corresponding to $\tilde{\Phi}_0(x,y)$ in (V.8), the pre-buckling stress distribution in the plate is given, for $0 \leq y \leq b$, by

$$\begin{aligned} \sigma^0_{xx} \equiv \tilde{\Phi}_{0,yy} = E\alpha T_0 \left(\frac{y}{b} - \frac{1}{2}\right) \\ \times \left(B_1 \sin hR_1 \frac{x}{a} \sin R_2 \frac{x}{a} + B_2 \cos hR_1 \frac{x}{a} \cos R_2 \frac{x}{a} + 1\right) \end{aligned} \tag{V.10a}$$

$$\begin{aligned} \sigma^0_{yy} \equiv \tilde{\Phi}_{0,xx} = \frac{1}{12} E\alpha T_0 \left(1 - 3\frac{y^2}{b^2} + 2\frac{y^3}{b^3}\right) \\ \times \left(D_1 \sin hR_1 \frac{x}{a} \sin R_2 \frac{x}{a} + D_2 \cos hR_1 \frac{x}{a} \cos R_2 \frac{x}{a}\right) \end{aligned} \tag{V.10b}$$

and

$$\begin{aligned} \sigma^0_{xy} \equiv -\tilde{\Phi}_{0,xy} = \frac{1}{2} E\alpha T_0 \left(1 - \frac{y}{b}\right) \frac{y}{b} \\ \times \left(D_3 \sin hR_1 \frac{x}{a} \cos R_2 \frac{x}{a} + D_4 \cos hR_1 \frac{x}{a} \sin R_2 \frac{x}{a}\right) \end{aligned} \tag{IV.10c}$$

with the stress components in the domain $-b \leq y \leq 0$ identical with those given by (V.10a,b,c). In (V.10a,b,c) the $D_i's$ are given by

$$\begin{cases} D_1 = B_1(k_1^2 - K_2^2) - 2B_2 k_1 k_2 \\ D_2 = B_2(k_1^2 - k_2^2) + 2B_1 k_1 k_2 \\ D_3 = B_1 k_2 + B_2 k_1 \\ D_4 = B_1 k_1 - B_2 k_2 \end{cases} \tag{V.11}$$

The stresses in the plate are a function of the temperature differential T_0 for the tentlike temperature distribution depicted in Fig. 11b and are independent of the edge temperature

T_1; as this temperature differential T_0 increases a value $(T_0)_{cr}$ or will be achieved at which the plate will buckle under the action of the induced thermal stresses. When only small deflection theory is used it may be assumed that the middle surface of the plate does not stretch and, thus, the stress distribution in the plate does not change after the onset of buckling; this stress distribution is given by (V.10a,b,c) while the corresponding plate deflection is governed by (I.21), with $\Delta M_{HT} = 0$ (as the temperature distribution is constant through the plate thickness) and $\Phi = \Phi_0$, i.e., by

$$K\Delta^2 w = h\left(\sigma_{xx}^0 w_{,xx} + 2\sigma_{xy}^0 w_{,xy} + \sigma_{yy}^0 w_{,yy}\right) \tag{V.12}$$

Of course, (V.12) also serves to determine the critical temperature differential and the corresponding first buckling mode within the context of large deflection theory; such a determination, in this case, may be achieved through the Rayleigh-Ritz method. Assuming a buckle pattern which is symmetrical about the center of the plate of the form

$$w = \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} a_{mn} \cos\frac{m\pi x}{2a} \cos\frac{n\pi y}{2b}, \tag{V.13}$$

substituting (V.13) and (V.10a,b,c) into the potential energy expression

$$\begin{aligned} U = \frac{1}{2}K \int_{-b}^{b}\int_{-a}^{a} &\left\{(\Delta w)^2 - 2(1-\nu)[w_{,xx}w_{,yy} - w_{,xy}^2]\right\} dxdy \\ +\frac{1}{2}h \int_{-b}^{b}\int_{-a}^{a} &\left[\sigma_{xx}^0 w_{,x}^2 + \sigma_{yy}^0 w_{,y}^2 + 2\sigma_{xy}^0 w_{,x} w_{,y}\right] dxdy \end{aligned}$$

and then minimizing with respect to the unknown coefficients a_{mn}, leads to a set of simultaneous equations, constituting a characteristic value problem, of the form

$$\frac{\pi^2 K}{b^2 E\alpha(T_0)_{cr} h} K_{pq} a_{pq} + \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} K_{pqmn} a_{mn} = 0 \tag{V.14}$$

$$p = 1, 3, 5...., q = 1, 3, 5, ...$$

The solutions of (V.14) yield sets of relative values of the coefficients a_{mn}, and associated values of the critical temperature $(T_0)_{cr}$; the expressions for the coefficients k_{pq}, k_{pqmn} in (V.14) are quite involved and are delineated in Appendix A of [2].

If only the terms a_{11}, a_{13}, a_{31}, and a_{33} are retained in the deflection function (V.13) then equations (V.14) can be written, for a plate having an aspect ratio of 1.57, in the matrix form

$$\begin{bmatrix} 15.73 & 22.52 & 14.88 & -5.96 \\ 0.504 & 1.426 & 0.871 & 0.377 \\ 1.35 & 3.54 & 7.42 & 7.79 \\ -0.0735 & 0.208 & 1.043 & 0.437 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{13} \\ a_{31} \\ a_{33} \end{bmatrix} = \frac{100\pi^2 K}{b^2 E\alpha (T_0)_{cr} h} \begin{bmatrix} a_{11} \\ a_{13} \\ a_{31} \\ a_{33} \end{bmatrix} \tag{V.15}$$

The solution of (V.15), for the smallest value of

$$\lambda = \frac{\pi^2 K}{b^2 E\alpha (T_0)_{cr} h},$$

is obtained by matrix iteration and yields $\lambda_{cr} = 5.39$. The relative values of the four coefficients which have been retained in the deflection function w are given by

$$\begin{bmatrix} a_{11} \\ a_{13} \\ a_{31} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 0.0365 \\ 0.1360 \\ 0.0042 \end{bmatrix}$$

in which case

$$\begin{aligned} w_0 = a_{11}\Big(\cos\frac{\pi x}{2a}\cos\frac{\pi y}{2b} + 0.0365\cos\frac{\pi x}{2a}\cos\frac{3\pi y}{2b} \\ +0.1360\cos\frac{3\pi x}{2a}\cos\frac{\pi y}{2b} + 0.0042\cos\frac{3\pi x}{2a}\cos\frac{3\pi y}{2b}\Big) \end{aligned} \tag{V.16}$$

It is easily computed, based on (V.16) that the deflection w_c at the center of the rectangular plate is $w_c = 1.1767\, a_{11}$ in which case

$$\begin{aligned} w_0 = \frac{w_c}{1.1767}\Big(\cos\frac{\pi x}{2a}\cos\frac{\pi y}{2b} + 0.0365\cos\frac{\pi x}{2a}\cos\frac{3\pi y}{2b} \\ +0.1360\cos\frac{3\pi x}{2a}\cos\frac{\pi y}{2b} + 0.0042\cos\frac{3\pi x}{2a}\cos\frac{3\pi y}{2b}\Big) \end{aligned} \tag{V.17}$$

In Table 2 we have indicated the convergence of the critical temperature parameter as additional terms are included in the deflection function; as indicated in [2], retaining further terms in the deflection function beyond those four already chosen, above, has a negligible effect on both the critical temperature and the buckle pattern. The initial buckling pattern ($\equiv$ small-deflection buckle pattern) in one quadrant of the plate is depicted in Fig. 13 for a plate with an aspect ratio of 1.57.

The postbuckling behavior of the heated plate which is analyzed in [2] takes into account, as all analyses of postbuckling behavior must, the stretching of the plate middle surface due to bending and the corresponding changes in the plate stress distribution as the plate deflects; thus, this analysis is based on the generalized von Karman equations for the hygrothermal case, i.e., (III.4a,b) with $\Delta M^T = 0$, $t = 0$. Actually, since the analysis in [2] uses the Airy function $\tilde{\Phi}$ based on the local stress distribution, the relevant form of (III.4a,b) in this case is

$$\left\{ \begin{aligned} K\Delta^2 w &= h\left(\tilde{\Phi}_{,yy}w_{,xx} + \tilde{\Phi}_{,xx}w_{,yy} - 2\tilde{\Phi}_{,xy}w_{,xy}\right) \\ \Delta^2\tilde{\Phi} &= -E\alpha \bigtriangledown^2 T + E\left[w_{,xy}^2 - w_{,xx}w_{,yy}\right] \end{aligned} \right. \tag{V.18}$$

where the lateral loading $t \equiv 0$.

The system (V.18) is solved (approximately) in [2] by using a procedure based on the Galerkin method. The stress function $\tilde{\Phi}$ is decomposed as the sum

$$\tilde{\Phi} = \tilde{\Phi}_0 + \tilde{\Phi}_1 \tag{V.19}$$

where $\tilde{\Phi}_0$ is the thermal stress (Airy) function for the unbuckled plate; $\tilde{\Phi}_0$ satisfies

$$\Delta^2\tilde{\Phi}_0 = -E_\alpha \bigtriangledown^2 T \tag{V.20}$$

and the stress boundary conditions and is given by (V.8), (V.9a,b,c). The function $\tilde{\Phi}_1$ is taken to be the solution of

$$\Delta^2\tilde{\Phi}_1 = E\left[w_{,xy}^2 - w_{,xx}w_{,yy}\right], \tag{V.21}$$

with w the buckle pattern determined by (V.13) and the boundary conditions on the stresses. For $\tilde{\Phi}_1$ a series of the form

$$\tilde{\Phi}_1 = (x^2 - a^2)^2(y^2 - b^2)^2(c_1 + c_2x^2 + c_3y^2 + ...) \tag{V.22}$$

is chosen and the coefficients $e_i,\ i = 1, 2, ...,$ are then determined in terms of the coefficients a_{mn} in (V.13) by the equations

$$\int_{-a}^{a}\int_{-b}^{b} \frac{\partial \tilde{\Phi}_1}{\partial c_i} \left\{ \Delta^2 \tilde{\Phi}_1 - E\left[w_{,xy}^2 - w_{,xx}w_{,yy} \right] \right\} dxdy = 0 \tag{V.23}$$

The resulting stress function $\tilde{\Phi}$ is now substituted into the first equation in (V.18) and the Galerkin approach is again used so as to determine the values of the coefficients a_{mn} of the deflection function w; as shown in [2], this leads to the set of simultaneous equations

$$\int_{-a}^{a}\int_{-b}^{b} \cos\frac{m\pi x}{a}\cos\frac{n\pi y}{2b}\Big(\frac{K}{n}\Delta^2 w - \tilde{\Phi}_{,yy}w_{,xx} - \tilde{\Phi}_{,xx}w_{,yy} + 2\tilde{\Phi}_{,xy}w_{,xy}\Big)dxdy = 0 \tag{V.24}$$
$$(m = 1, 3, 5, ...,\ n = 1, 3, 5, ...)$$

for the a_{mn}. The equations (V.24) are, of course, nonlinear and their solution becomes more difficult as the number of terms retained in the deflection function increases. In [2], therefore, it is assumed that the shape of the deflected surface of the plate for large deflections may be taken to be the one the plate has at the onset of buckling; with such an assumption only the coefficient a_{11} remains arbitrary while the ratios a_{mn}/a_{11} are taken to be those which are given by the initial buckling solution described above. In lieu of (V.24) we have, therefore, as the Galerkin-type equation, from which the coefficient a_{11} may be determined, the relation

$$\int_{-a}^{a}\int_{-b}^{b} w_s\Big(\frac{K}{h}\Delta^2 w_s - \tilde{\Phi}_{,yy}w_{s,xx} - \tilde{\Phi}_{,xx}w_{s,yy} + 2\tilde{\Phi}_{,xy}w_{s,xy}\Big)dxdy = 0 \tag{V.25}$$

where

$$w_s = \sum_{m=1,3,5}^{\infty}\sum_{n=1,3,5}^{\infty} \frac{a_{mn}}{a_{11}}\cos\frac{m\pi x}{2a}\cos\frac{n\pi y}{2b} \tag{V.26}$$

In (V.25) the ratios a_{mn}/a_{11} which were obtained from the initial buckling solution must also be substituted into the stress function $\tilde{\Phi}$.

For a plate with aspect ratio $a/b = 1.57$ it has already been determined that

$$w_s = \cos\frac{\pi x}{2a}\cos\frac{\pi y}{2b} + 0.0365\cos\frac{\pi x}{2a}\cos\frac{3\pi y}{2b} + 0.1360\cos\frac{3\pi x}{2a}\cos\frac{\pi y}{2b} + 0.0042\cos\frac{3\pi x}{2a}\cos\frac{3\pi y}{2b}$$

in which case (V.25) yields the following relation between the temperature differential T_0 and the center deflection w_c:

$$\frac{b^2 E\alpha T_0 h}{\pi^2 K} = 5.39 + 1.12(1-\nu^2)\frac{w_c^2}{h^2} \tag{V.27}$$

At any other point of the plate (assumed to have stress-free edges and an aspect ratio $a/b = 1.57$) the deflection assumes the form

$$\frac{w}{h} = 0.723\sqrt{\frac{\frac{b^2 E\alpha T_0 h}{\pi^2 K} - 5.39}{1.12(1-\nu^2)}}\, w_s \tag{V.28}$$

A comparison of the calculated deflections at the plate center with some experimental data is shown in Fig. 14 while in Figs. 15 a,b, respectively, we depict the predicted growth of the deflections, with increasing temperature differential T_0, along the longitudinal center line and the transverse center line of the rectangular plate. We will return to the problem treated in [2] when we discuss hygrothermal buckling in the presence of imperfections in Chapter VII.

As a final example of hygrothermal buckling of heated, isotropic, rectangular plates we consider the large deflection analysis presented in [20]; in this paper the plate is subjected to both heating and resultant edge loads due to an elastic edge restraint in the plane of the plate. A major result of the analysis presented in [20] is that a decreased buckling temperature results from a lowered edge flexibility. The temperature gradient in the z-direction through the thickness of the plate is assumed, in [20], to be negligible and the edges of the plate are assumed to remain straight during deformation. The rectangular plate, as depicted in Fig. 16, is simply supported along all four edges and free to rotate at the edges but translation of the edges in the plane of the plate is resisted by spring forces of magnitude k_y (force/cubic volume) along $y = 0$ and $y = b$ and k_x along $x = 0$ and $x = a$.

The relevant form of the large deflection equations for the problem at hand is, precisely, (V.18) where $\tilde{\Phi}$ is the Airy function associated with the local stress field $(\sigma_{xx}, \sigma_{xy}, \sigma_{yy})$ as opposed to the resultant stresses (N_x, N_{xy}, N_y). The boundary conditions require that the deflection w and the edge bending moments per unit length be zero along the edges of the plate; as $M^T = 0$, because of the assumed constancy of the temperature distribution through the thickness of the plate, these boundary conditions (see (III.7)) assume the form $w = 0$ along $x = 0, a$, for $0 \leq y \leq b$, $w = 0$ along $y = 0, b$, for $0 \leq x \leq a$, and

$$\begin{cases} w_{,xx} + \nu w_{,yy} = 0, \; x = 0, a; \; 0 \leq y \leq b \\ w_{,yy} + \nu w_{,xx} = 0, \; y = 0, b; \; 0 \leq x \leq a \end{cases} \tag{V.29}$$

The boundary conditions are satisfied by choosing

$$w = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} a_{rs} \sin \frac{r\pi x}{a} \sin \frac{s\pi y}{b} \tag{V.30}$$

The lateral loading $t(x, y)$ is represented as a Fourier series in the form

$$t = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} b_{rs} \sin \frac{r\pi x}{a} \sin \frac{s\pi y}{b} \tag{V.31}$$

while the prescribed temperature distribution is represented as

$$T = T_0 + \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} T_{pq} \cos \frac{p\pi x}{a} \cos \frac{q\pi y}{b} \tag{V.32}$$

with T_0 the average temperature over the surface of the plate. As the plate is subjected to the temperature distribution given by (V.32), the plate will expand and this expansion will result in a distribution of average edge thrusts which are caused by the restraining spring forces acting along the edges of the plate; we denote these average edge stresses by ρ and ξ, respectively, along the edges $x = 0$, a and $y = 0, b$. The magnitudes of ρ and ξ depend on the temperature distribution, the plate dimensions, and the elastic moduli of the plate. Employing (V.30) and (V.32) it may be shown that the second of the large deflection equations in (V.18) is satisfied if $\tilde{\Phi}$ has the form

$$\tilde{\Phi} = \frac{\xi x^2}{2} + \frac{\rho y^2}{2} + \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} c_{pq} \cos\frac{p\pi x}{a} \cos\frac{q\pi y}{b} \tag{V.33}$$

in which

$$c_{pq} = E\left\{\sum_{i=1}^{9} B_i + A_{pq}\right\} \Big/ 4\left(\frac{p^2 b}{a} + \frac{q^2 a}{b}\right)^2 \tag{V.34}$$

where

$$A_{pq} = \left(\frac{4\alpha}{\pi^2}\right) T_{pq} ab \left(\frac{p^2 b}{a} + \frac{a^2 a}{b}\right) \tag{V.35}$$

while the $B_i, i = 1, 2, \ldots 9$ are given in [21]; the work in [21] covers the case of a laterally loaded plate subject to edge thrusts in the plane of the plate with lateral displacements in the large-deflection range but does not consider temperature effects. Specific expressions for the coefficients b_{rs}, in terms of the coefficients a_{rs}, c_{rs}, so that the first equation in (V.18) is satisfied are quite complex; they are delineated in [20] and will not be replicated here.

For the plate to be in equilibrium it is necessary that

$$\int_0^a \left[\tilde{\Phi}_{,xx} - k_y v(x, y)\right] dx = 0 \text{ at } y = 0, b;\ \text{for } 0 \leq x \leq a \tag{V.36a}$$

and

$$\int_0^b \left[\tilde{\Phi}_{,yy} - k_x u(x, y)\right] dy = 0 \text{ at } x = 0, a;\ \text{for } 0 \leq y \leq b \tag{V.36b}$$

We recall the constitutive relations in the form

$$\begin{cases} \epsilon_{xx}^0 \equiv u_{,x} + \dfrac{1}{2} w_{,x}^2 = \dfrac{1}{E}\left(\tilde{\Phi}_{,yy} - \nu\tilde{\Phi}_{,xx} + \alpha E T\right) \\ \epsilon_{yy}^0 \equiv v_{,y} + \dfrac{1}{2} w_{,y}^2 = \dfrac{1}{E}\left(\tilde{\Phi}_{,xx} - \nu\tilde{\Phi}_{,yy} + \alpha E T\right) \end{cases} \tag{V.37}$$

As the spring constants on opposite edges of the plate are equal, it follows that for plate equilibrium to hold the displacements along opposite edges must also be equal in magnitude, i.e.,

$$\begin{cases} u(a, y) = -u(0, y),\ 0 \leq y \leq b \\ v(x, b) = -v(x, 0),\ 0 \leq x \leq a \end{cases} \tag{V.38}$$

Employing (V.38) in conjunction with (V.37) we obtain

$$\begin{cases} u(0,y) = -\dfrac{1}{2E}\displaystyle\int_0^a \left[\tilde{\Phi}_{,yy} - \nu\Phi_{,xx} + \alpha ET - \frac{E}{2}w_{,x}^2\right] dx \\ v(x,0) = -\dfrac{1}{2E}\displaystyle\int_0^b \left[\tilde{\Phi}_{,xx} - \nu\tilde{\Phi}_{,yy} + \alpha ET - \frac{E}{2}w_{,y}^2\right] dy \end{cases} \tag{V.39}$$

Carrying out the integrations in (V.39), and making use of (V.30)-(V.32), we find for the edge displacements in the plane of the plate

$$\begin{aligned} u(0,y) = -\frac{1}{2E}\Bigg[&\rho a - \nu\xi a + \alpha ET_0 a \\ &-\frac{aE}{4}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\sum_{s'=1}^{\infty} a_{rs}a_{rs'}\left(\frac{r\pi}{a}\right)^2 \sin\frac{s\pi y}{b}\sin\frac{s'\pi y}{b}\Bigg] \end{aligned} \tag{V.40a}$$

and

$$\begin{aligned} v(x,0) = -\frac{1}{2E}\Bigg[&\xi b - \nu\rho b + \alpha ET_0 b \\ &-\frac{bE}{4}\sum_{r=1}^{\infty}\sum_{r'=1}^{\infty}\sum_{s=1}^{\infty} a_{rs}a_{rs'}\left(\frac{s\pi}{b}\right)^2 \sin\frac{r\pi x}{a}\sin\frac{r'\pi x}{a}\Bigg] \end{aligned} \tag{V.40b}$$

Substituting (V.33) and (V.40a,b) into (V.36a,b) and carrying out the indicated integrations we obtain

$$\begin{cases} \rho\lambda_{11} + \xi\lambda_{12} = \mu_{11} \\ \rho\lambda_{21} + \xi\lambda_{22} = \mu_{22} \end{cases} \tag{V.41}$$

where

$$\begin{cases} \lambda_{11} = -\dfrac{\nu k_y b}{2E}, \ \lambda_{12} = 1 + \dfrac{k_y b}{2E} \\ \lambda_{22} = -\dfrac{\nu k_x a}{2E}, \ \lambda_{22} = 1 + \dfrac{k_x a}{2E} \end{cases} \tag{V.42a}$$

$$\begin{cases} \mu_{11} = -\dfrac{k_y b}{2E}\left[\alpha ET_0 - \dfrac{1}{8}E\displaystyle\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} a_{rs}^2\left(\frac{s\pi}{b}\right)^2\right] \\ \mu_{22} = -\dfrac{k_x a}{2E}\left[\alpha ET_0 - \dfrac{1}{8}E\displaystyle\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} a_{rs}^2\left(\frac{r\pi}{a}\right)^2\right] \end{cases} \tag{V.42b}$$

From (V.41) we obtain the average edge (thrust) stresses in the form

$$\rho = (\lambda_{22}\mu_{11} - \lambda_{12}\mu_{22})/(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}) \tag{V.43a}$$

and

$$\xi = (\lambda_{11}\mu_{22} - \lambda_{21}\mu_{11})/(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}) \tag{V.43b}$$

An expression for the (critical) buckling temperature can now be derived in terms of the physical properties of the plate, the plate dimensions, and the spring constants k_x and k_y. In [20], only the case of a uniform temperature distribution $T = T_0$ is considered; we assume, also, that the lateral loading $t \equiv 0$ so that $b_{rs} \equiv 0$. As $(T_0)_{cr}$ corresponds to the first buckling mode the summation indices r, s in (V.30) are both equal to one when $T_0 = (T_0)_{cr}$. For initial buckling, $w = 0$ in the second equation in (V.18) in which case $\tilde{\Phi}$ as defined by (V.33) reduces to

$$\tilde{\Phi}_0 = \frac{\xi x^2}{2} + \frac{\rho y^2}{2} \tag{V.44}$$

and the first equation in (V.18) reduces to the statement that

$$\frac{E\pi^2}{12(1-\nu^2)} \left(\frac{h}{a}\right)^2 \left[1 + \left(\frac{a}{b}\right)^2\right] = -\rho - \left(\frac{a}{b}\right)^2 \xi \tag{V.45}$$

Using (V.42a,b), (V.43a,b) in (V.45), with the summations extending only over $r = s = 1$, then yields the following expression for the buckling temperature

$$\begin{cases} (\alpha T_0)_{cr} = \hat{P}/\hat{Q} \\ \hat{P} = \dfrac{\pi^2}{12(1-\nu^2)} \left(\dfrac{h}{a}\right)^2 \left[1 + \left(\dfrac{a}{b}\right)^2\right] \left[\left(1 + \dfrac{2E}{k_x a}\right)\left(1 + \dfrac{2E}{k_y b}\right) - \nu^2\right] \\ \hat{Q} = \left[1 + \left(\dfrac{a}{b}\right)^2\right](1+\nu) + \left(\dfrac{2E}{k_y b}\right) + \left(\dfrac{a}{b}\right)^2 \left(\dfrac{2E}{k_x a}\right) \end{cases} \tag{V.46}$$

If $k_x = k_y = \infty$ then (V.46) yields the expression for the buckling temperature for the case of a nonflexible edge restraint in the form

$$(\alpha T_0)_{cr}^{\infty} = \frac{\pi^2}{12(1+\nu)} \left(\frac{h}{a}\right)^2 \left[1 + \left(\frac{a}{b}\right)^2\right] \tag{V.47}$$

in which case, for $a = b$ and $k_x = k_y$

$$\frac{(\alpha T_0)_{cr}}{(\alpha T_0)_{cr}^{\infty}} = \frac{1}{1-\nu} \cdot \frac{\left(1 + \dfrac{2E}{k_x a}\right)^2 - \nu^2}{1 + \nu + \dfrac{2E}{k_x a}} \tag{V.48}$$

Numerical results have been generated by the authors in [20] for the case of a uniform temperature distribution $T = T_0$ and a sinusoidal lateral loading of the form

$$t = b_{11} \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \tag{V.49}$$

Plots of the center deflection of the plate versus a lateral load parameter based on q_{11} are depicted in Fig. 18 for various values of T_0, the aspect ratio a/b, and the spring constant $k_x (k_x = k_y$ in the application). For a uniform lateral loading the variation of the buckling temperature ratio $(\alpha T_0)_{cr}/(\alpha T_0)_{cr}^{\infty}$ with the edge spring rate parameter $k_x a/2E$ is depicted in Fig. 18 ($a = b$, $k_x = k_y$ in the application) while for the same loading condition the effect of the edge fixity on the critical temperature is displayed in Fig. 19; finally, the center deflection versus temperature ratio (postbuckling) curves for this case are depicted in Fig. 20 for several different values of $k_x a/2E$.

Having discussed several examples of buckling and postbuckling behavior for heated, isotropic, rectangular plates we now turn to the rectilinear orthotropic case for such a geometry. For this problem the relevant generalized von Karman equations are given by (III.14a,b) where we set the applied transverse loading $t(x, y) \equiv 0$. Athough many of the papers in the literature, which deal with the buckling behavior of heated, rectangular, orthotropic plates, do so within the context of the Berger's approximation, which will be discussed in the next Chapter, there are a few analyses which deal with the full system (III.14a,b) in the context of plate bending, e.g., the work of Biswas [22]; this work appears to contain a serious flaw, common to much of the literature on thermal bending and buckling of simply supported plates, as we shall indicate below.

In [22] the author writes (III.14a), with $t \equiv 0$, in the form

$$\begin{aligned} D_x w_{,xxxx} + 2H w_{,xxyy} + D_y w_{,yyyy} \\ + \tilde{\beta}_1 M_{T,xx} + \tilde{\beta}_2 M_{T,yy} = \\ F_{,xx} w_{,yy} - 2F_{,xy} w_{,xy} + F_{,yy} w_{,xx} \end{aligned} \tag{V.50}$$

$$M_T = \int_{-h/2}^{h/2} \delta T(x,y,z) z dz$$

and it is easy to see that, with respect to the notation employed in the present work

$$D_x = D_{11},\ 2H = D_{12} + 4D_{66} + D_{21}, D_y = D_{22} \tag{V.51}$$

while $F = \Phi$ and

$$\begin{cases} \tilde{\beta}_1 = c_{11}\alpha_1 + c_{12}\alpha_2 \\ \tilde{\beta}_2 = c_{21}\alpha_1 + c_{22}\alpha_2 \end{cases} \tag{V.52}$$

The $\tilde{\beta}_i$ in (V.52) are not to be confused with their previous interpretation as hygroscopic expansion coefficients (where we denoted them as β_i). In order to bring the second von Karman equation for this case into line with the form employed in [22] we multiply (III.14b) through by $E_2 h$ thus producing

$$\begin{aligned} \Phi_{,xxxx} &+ \left(\frac{E_2}{G_{12}} - 2\nu_{12}\right)\Phi_{,xxyy} + \left(\frac{E_2}{E_1}\right)\Phi_{,yyyy} \\ &= -\frac{1}{2}E_2 h[w,w] - (\tilde{N}_T^2 - \nu_{12}\tilde{N}_T^1)_{,xx} \\ &- \left(\frac{E_2}{E_1}\right)(\tilde{N}_T^1 - \nu_{21}\tilde{N}_T^2)_{,yy} \end{aligned} \tag{V.53}$$

Next, we note that, by virtue of (III.11),

$$\tilde{N}_T^2 - \nu_{12}\tilde{N}_T^1 = \left[\left(\frac{E_1\alpha_1\nu_{12} + E_2\alpha_2\nu_{21}}{1-\nu_{12}\nu_{21}}\right) - \nu_{21}\left(\frac{E_1\alpha_1 + E_2\alpha_2\nu_{21}}{1-\nu_{12}\nu_{21}}\right)\right] N_T \equiv E_2\alpha_2 N_T \tag{V.54a}$$

while

$$\tilde{N}_T^2 - \nu_{12}\tilde{N}_T^2 = \left[\left(\frac{E_1\alpha_1 + E_2\alpha_2\nu_{21}}{1-\nu_{12}\nu_{21}}\right) - \nu_{21}\left(\frac{E_1\alpha_1\nu_{12} + E_2\alpha_2}{1-\nu_{12}\nu_{21}}\right)\right] N_T \equiv E_1\alpha_1 N_T \tag{V.54b}$$

where

$$N_T = \int_{-h/2}^{h/2} \delta T(x,y,z) dz$$

In view of (V.54a,b), (V.53) becomes

$$\begin{aligned} \Phi_{,xxxx} &+ \left(\frac{E_2}{G_{12}} - 2\nu_{12}\right)\Phi_{,xxyy} + \left(\frac{E_2}{E_1}\right)\Phi_{,yyyy} \\ &= E_2 h(w_{,xy}^2 - w_{,xx}w_{,yy}) - E_2\alpha_2 N_{T,xx} - E_2\alpha_1 N_{T,yy} \end{aligned} \tag{V.55}$$

Equation (V.55) corresponds to the second generalized von Karman equation (for this case) in [22], i.e., to

$$F_{,xxxx} + p^2 F_{,xxyy} + q^2 F_{,yyyy} + \lambda_1 N_{T,xx} + \lambda_2 N_{T,yy} = E_2 h(w_{,xy}^2 - w_{,xx} w_{,xx} w_{,yy}) \tag{V.56}$$

if we again identify $F = \Phi$ and take

$$\begin{cases} p^2 = \dfrac{E_2}{G_{12}} - 2\nu_{12} \\ q^2 = E_2/E_1 \\ \lambda_1 = E_2\alpha_2 \\ \lambda_2 = E_2\alpha_1 \end{cases} \tag{V.57}$$

With the correlations $F = \Phi$, (V.51), (V.57)), and the definitions of N_T, M_T, given above, (V.50) and (V.56) are identical with our earlier equations for the heated, rectangular orthotropic plate, namely, (IV.14a,b).

Remarks: Biswas [22] writes the λ_i of (V.57) in the form

$$\begin{cases} \lambda_1 = \dfrac{E_2}{E_1}\tilde{\beta}_1 - \nu_{12}\tilde{\beta}_2 \\ \lambda_2 = \tilde{\beta}_2 - \nu_{12}\tilde{\beta}_1 \end{cases} \tag{V.58}$$

It is easily verified, by using (V.52), that these expressions are equivalent to those in (V.57). We also note that Biswas [22] works with the inverse constitutive relations, in lieu of (I.25), i.e. with the matrix $s_{ij} = c_{ij}^{-1}$ whose components are related to the c_{ij} by

$$\begin{cases} c_{11} = s_{22}/\Delta \\ c_{12} = -s_{12}/\Delta \\ c_{21} = s_{11}/\Delta \\ c_{22} = -s_{12}/\Delta \\ c_{66} = 1/s_{66} \end{cases} \tag{V.59}$$

with $\Delta = s_{11}s_{22} - s_{12}^2$. It is any easy task to show that, based on (I.27), (V.59),

$$\begin{cases} E_1 = \dfrac{1}{s_{11}}, \; E_2 = \dfrac{1}{s_{22}}, \; \nu_{12} = -\dfrac{s_{12}}{s_{11}} \\ \nu_{21} = -\dfrac{s_{12}}{s_{22}}, \; G_{12} = \dfrac{1}{s_{66}} \end{cases} \tag{V.60}$$

We now consider a simply supported orthotropic plate which occupies the region $0 \leq x \leq a$, $0 \leq y \leq b$, $-\frac{1}{2}h \leq z \leq \frac{1}{2}h$ in the three-space and assume a temperature distribution which varies linearly through the plate of the form

$$\delta T(x, y, z) = \frac{1}{2}(T_1 + T_2) + z\left(\frac{T_1 - T_2}{h}\right) \tag{V.61}$$

so that

$$T(x, y, \frac{1}{2}h) = T_1, \; T(x, y, -\frac{1}{2}h) = T_2 \tag{V.62}$$

Corresponding to the simple-support conditions, along the edges of the plate, the author, in [22] looks for plate deflections of the form

$$w(x, y) = w_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \tag{V.63}$$

However, by virtue of (III.15), the assumption (V.63) cannot possibly be valid unless $\tilde{M}_T^1 = \tilde{M}_T^2 = 0$, i.e., unless $M_T = 0$ (which is not true in view of (V.61)). For the temperature distribution (V.61), $N_{T,x} = N_{T,y} = 0$; Using (V.63) in (V.56) it may be shown that the general solution of (V.56) is given by an Airy function $F(x, y)$ of the form

$$F = \frac{1}{2}Ax^2 + \frac{1}{2}By^2 + \frac{1}{32}E_2 h w_0^2 \left(\frac{a^2}{b^2}\cos\frac{2\pi x}{a} + \frac{b^2}{a^2 q^2}\cos\frac{2\pi y}{b}\right) \tag{V.64}$$

with A and B arbitrary constants that must be determined from the in-plane boundary conditions. We assume that all edges of the plate remain straight after deformation; then, using the strain-displacement equations, the constitutive relations (III.10) for the resultant stresses, (V.64), and the definitions (I.27), (V.59) of the constitutive constants one obtains for the in-plane displacements

$$u = \int_0^a \left[\frac{s_{11}}{h} \left(B - \frac{E_2 h w_0^2 \pi^2}{8a^2 q^2} \cos \frac{2\pi y}{b} \right) \right.$$

$$+ \frac{s_{12}}{h} \left(A - \frac{E_2 h w_0^2 \pi^2}{8b^2} \cos \frac{2\pi x}{a} \right) \tag{V.65a}$$

$$\left. - \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{\tilde{\beta}_1 s_{11} + \tilde{\beta}_2 s_{12}}{h} N_T \right]_{y=0} dx$$

$$v = \int_0^b \left[\frac{s_{22}}{h} \left(A - \frac{E_2 h w_0^2 \pi^2}{8b^2} \cos \frac{2\pi x}{a} \right) \right.$$

$$+ \frac{s_{12}}{h} \left(B - \frac{E_2 h w_0^2 \pi^2}{8a^2 q^2} \cos \frac{2\pi y}{b} \right) \tag{V.65b}$$

$$\left. - \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \frac{\tilde{\beta}_1 s_{12} + \tilde{\beta}_2 s_{22}}{h} N_T \right]_{x=0} dy$$

For immovable plate edges (in the plane of the plate) $u = v = 0$; combining this latter assumption with (V.65a,b) we are led to

$$A = C_1 w_0^2 - \tilde{\beta}_2 N_T, \; B = C_2 w_0^2 \tilde{\beta}_1 N_T \tag{V.66a}$$

with

$$\begin{cases} C_1 = \dfrac{E_2 h \pi^2}{8\Delta} \left(\dfrac{s_{11} s_{22}}{a^2 q^2} - \dfrac{s_{22} s_{11}}{b^2} \right) \\ C_2 = \dfrac{E_2 h \pi^2}{8\Delta} \left(\dfrac{s_{11} s_{22}}{a^2 q^2} - \dfrac{s_{22} s_{12}}{b^2} \right) \end{cases} \tag{V.66b}$$

Finally, an algebraic equation for w_0/h is determined in [22] by using (V.63), (V.64), (V.66a,b) and the Fourier expansion

$$M_T = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{16 M_T}{mn\pi^2} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{V.67}$$

in (V.50); this procedure yields, in [22], the following first-order approximation

$$\frac{w_0}{h} = \frac{16(M_T/h)(\tilde{\beta}_1 a^{-2} + \tilde{\beta}_2 b^{-2})}{\pi^4[(D_x/a^4) + (2H/a^2b^2) + (Dy/b^4)]} \tag{V.68}$$

We note, however, the objection to the analysis which has been raised following (V.63) as well as the observation that the double Fourier series for M_T, as given by (V.67), formally vanishes for $x = 0, a$ $(0 \leq y \leq b)$ and $y = 0, b$ $(0 \leq x \leq a)$ whereas, in point of fact

$$M_T \equiv \int_{-n/2}^{h/2} \delta T(x,y,z) z dz = \frac{1}{12}(T_1 - T_2)h^2 \neq 0$$

if $T_1 \neq T_2$. Variations of the non-dimensional (bending) deflections, as a function of the temperature parameter $\dfrac{\tilde{\beta}_2}{10\tilde{\beta}_1}(T_1 - T_2)$, which are based on (V.68), are depicted in Fig. 21. We now comment, below, on what appears to be the central shortcoming in the work presented in [22].

Remarks: For a simply supported, rectangular, orthotropic elastic plate occupying the domain $0 \leq x \leq a$, $0 \leq y \leq b$, $-\dfrac{h}{2} \leq z \leq \dfrac{h}{2}$, the relevant boundary conditions are given by (see (III.13), (III.15):

$$\left\{ \begin{array}{c} D_{11} w_{,xx} + D_{12} w_{,yy} + \tilde{M}_T^1 = 0 \\ w = 0 \end{array} \right. \tag{V.69a}$$

for $x = 0, a,$ $0 \leq y \leq b$, and

$$\left\{ \begin{array}{c} D_{21} w_{,xx} + D_{22} w_{,yy} + \tilde{M}_T^2 = 0 \\ w = 0 \end{array} \right. \tag{V.69b}$$

for $y = 0, b, 0 \leq x \leq a$, where

$$\tilde{M}_T^2 = (c_{11}\alpha_1 + c_{12}\alpha_2) \int_{-h/2}^{h/2} \delta T(x,y,z) dz \tag{V.70a}$$

$$\tilde{M}_T^2 = (c_{21}\alpha_1 + c_{22}\alpha_2) \int_{-h/2}^{h/2} \delta T(x,y,z) dz \tag{V.70b}$$

From the assumed from of the temperature distribution in Biswas [22], i.e.,

$$\delta T(x,y,z) = \frac{1}{2}(T_1 + T_2) + z\left(\frac{T_1 - T_2}{h}\right),$$

T_1 and T_2 being, respectively, the constant temperatures at the top and bottom plate surfaces,

$$\int_{-h/2}^{h/2} \delta T(x,y,z) dz = \frac{1}{2} h (T_1 + T_2)$$

so that

$$\begin{cases} \tilde{M}_T^1 = \dfrac{(c_{11}\alpha_1 + c_{12}\alpha_2)}{2} h(T_1 + T_2) \equiv A^* \\ \tilde{M}_T^2 = \dfrac{(c_{21}\alpha_1 + c_{22}\alpha_2)}{2} h(T_1 + T_2) \equiv B^* \end{cases} \tag{V.71}$$

Setting $D_{11} = \lambda_1,\ D_{12} = \lambda_2,\ D_{21} = \gamma_1,\ D_{22} = \gamma_2$, the boundary conditions of simple support which are represented by (V.69a,b), (V.70a,b) become

$$\begin{cases} w(0, y) = w(a, y) = 0 \\ \lambda_1 w_{,xx}(0, y) + \lambda_2 w_{,yy}(0, y) = A^* \\ \lambda_1 w_{,xx}(a, y) + \lambda_2 w_{,yy}(a, y) = A^* \end{cases} \tag{V.72a}$$

for $0 \leq y \leq b$ and

$$\begin{cases} w(x, 0) = w(x, b) = 0 \\ \gamma_1 w_{,xx}(x, 0) + \gamma_2 w_{,yy}(x, 0) = B^* \\ \gamma_1 w_{,xx}(x, b) + \gamma_2 w_{,yy}(x, b) = B^* \end{cases} \tag{V.72b}$$

for $0 \leq x \leq a$. However, as $w(0, y) = 0,\ 0 \leq y \leq b$, it follows that $w_{,yy}(0, y) = 0,\ 0 \leq y \leq b$, and, similarly, $w(a, y) = 0,\ 0 \leq y \leq b$, implies that $w_{,yy}(a, y) = 0,\ 0 \leq y \leq b$. In an analogous manner we obtain from $w(x, 0) = 0,\ 0 \leq x \leq a$, the fact that $w_{,xx}(x, 0) = 0$ $0 \leq x \leq a$, and from $w(x, b) = 0,\ 0 \leq x \leq a$, the conclusion that $w_{,xx}(x, b) = 0, 0 \leq x \leq a$. Thus, (V.72a), (V.72b) imply that along the edges of the plate we must have

$$\begin{cases} w(0, y) = w(a, y) = 0,\ 0 \leq y \leq b \\ w(x, 0) = w(x, b) = 0,\ 0 \leq x \leq a \end{cases} \tag{V.73a}$$

$$\begin{cases} w_{,xx}(0, y) = w_{,xx}(a, y) = A^*/\lambda_1,\ 0 \leq y \leq b \\ w_{,yy}(x, 0) = w_{,yy}(x, b) = B^*/\gamma_2,\ 0 \leq x \leq a \end{cases} \tag{V.73b}$$

The situation is depicted in Fig. 22. In [22], Biswas, as we have already indicated, above, looks for plate deflections of the form specified in (V.63), i.e.,

$$w(x, y) = w_0 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \tag{V.74}$$

However, such deflections can satisfy (V.73a,b) if and only if $A^* = B^* = 0$, i.e., if and only if $T_1 = T_2 = 0$. If, on the other hand, the boundary conditions of simple support are to be compatable with an applied temperature distribution which is independent of position in the middle surface of the plate, and varies linearly through the plate thickness, trial (bending) deflections would have to satisfy the inhomogeneous boundary data (V.73a,b); this is, clearly, not possible as should be evident from Fig. 22, because at each of the four corners of the plate the boundary data is inconsistent, e.g., $w_{,xx} = A^*/\lambda_1 \neq 0$, for $x = 0$, $0 \leq y \leq b$, which implies that $w_{,xx}(0,0) = A^*/\lambda_1$. However, $w = 0$, for $y = 0$, $0 \leq x \leq a$, so that $w_{,xx}(x,0) = 0$, $0 \leq x \leq a$, in which case $w_{,xx}(0,0) = 0$. Thus any (trial) deflection which satisfied the inhomogeneous boundary conditions (V.73a,b) would have to exhibit jump discontinuities in its second derivatives at each of the four corners of the rectangular plate. Such a problem would not exist for an applied temperature distribution $\delta T(x,y,z)$ which produced nontrivial thermal moments $\tilde{M}_T^1$, $\tilde{M}_T^2$ satisfying $\tilde{M}_T^1(0,0) = \tilde{M}_T^1(0,b) = \tilde{M}_T^1(a,0) = \tilde{M}_T^1(a,b) = 0$ and $\tilde{M}_T^2(0,0) = \tilde{M}_T^2(a,0) = \tilde{M}_T^2(0,b) = \tilde{M}_T^2(a,b) = 0$; such thermal moments could still satisfy

$$\begin{cases} \dfrac{\partial \tilde{M}_T^1}{\partial y}(0,y) \neq 0, \ \dfrac{\partial \tilde{M}_T^1}{\partial y}(a,y) \neq 0, \ 0 \leq y \leq b \\ \dfrac{\partial \tilde{M}_T^2}{\partial x}(x,0) \neq 0, \ \dfrac{\partial \tilde{M}_T^2}{\partial x}(x,b) \neq 0, \ 0 \leq x \leq a \end{cases} \tag{V.75}$$

as long as each of $\tilde{M}_T^1, \tilde{M}_T^2$ vanished at the four corners of the plate. We now turn to a discussion of the initial buckling behavior of circular plates within the context of large deflection theory.

Although the interpretation is somewhat different, the analysis of initial buckling for an isotropic circular plate subject to thermal stresses, within the context of large deflection theory, mathematically parallels the small deflection analysis presented in the previous section. Klosner and Forray [13] studied the buckling of simply supported isotropic circular plates subjected to a symmetrical temperature distribution by using the Rayleigh-Ritz method. For the case of a circular plate of variable thickness, Mansfield [23] has analyzed the buckling, curling, and postbuckling behavior when the temperature varies through the thickness of the plate and the edge of the plate is restrained in its plane; we will discuss Mansfield's work in section VII and other discussions of the postbuckling behavior of isotropic circular plates will appear in section VI within the context of Berger's approximation. In this section we will content ourselves with discussions of the work of Sarkar [24] who treats the case of a heated thin circular plate of isotropic material under uniform thermal compression in the plane of the plate, and the analysis in Nowacki [8] which is based on the work of Vinokurov [25]; in [24], the critical buckling temperature is calculated for plates under different edge conditions and different temperature distributions as well. The edge of the plate is restrained in the plane of the plate.

The solid circular isotropic plate considered by Sarkar [24] has uniform thickness, occupies the domain $0 \leq r \leq a$, $-\frac{h}{2} \leq z \leq \frac{h}{2}$, and is subjected to an arbitrary temperature distribution (which, however, must be symmetrical in order for the author's [24] equations to apply). As the edge of the plate is assumed to be restrained in the plane of the plate, the displacement $u(r) = 0$. Setting $\phi = -\frac{dw}{dr}$ we have, as a direct consequence of (III.19), (III.21), the assumed radial symmetry of w, and the fact that $u(r) \equiv 0$, $0 \leq r \leq a$,

$$N_r = -N_T^* = -\frac{E\alpha}{1-\nu}\int_{-h/2}^{h/2} T(r,z)dz \tag{V.76}$$

and

$$\begin{cases} M_r = K\left[\dfrac{d\phi}{dr} + \dfrac{\nu}{r}\phi\right] - M_T^* \\ M_\theta = K\left[\nu\dfrac{d\phi}{dr} + \dfrac{1}{r}\phi\right] - M_T^* \end{cases} \tag{V.77}$$

where M_T^* is given by (III.22). With the assumptions given above, the initial buckling equation that follows from the system (III.23) by setting $w = 0$ in the second equation of this set, and substituting the resultant Airy function in the first equation, is

$$r^2\frac{d^2\phi}{dr^2} + r\frac{d\phi}{dr} + \left(\frac{N_r}{K}r^2 - 1\right)\phi = r^2\frac{dM_T^*}{dr} \tag{V.78}$$

Replacing $\dfrac{N_r}{K} \equiv -\dfrac{1}{K}N_T^*$ in (V.78) by γ^2 and then setting $v(r) = \gamma r$ we obtain the equation

$$\frac{d^2\phi}{dv^2} + \frac{1}{v}\frac{d\phi}{dv} + \left(1 - \frac{1}{v^2}\right)\phi = \frac{1}{\gamma}\frac{dM_T^*}{dv} \tag{V.79}$$

As a simple example in [24] the author considers the case of a temperature distribution which varies only through the thickness of the plate, i.e., $\delta T = \delta T(z)$. In this case, M_T^* is clearly independent of r, and hence of v, so that $\dfrac{dM_T^*}{d\nu} = 0$. Equation (V.79) then reduces to

$$\frac{d^2\phi}{dv^2} + \frac{1}{v}\frac{d\phi}{dv} + \left(1 - \frac{1}{v^2}\right)\phi = 0 \tag{V.80}$$

for $0 \leq v \leq \gamma a$. The solution of (V.80) which is finite at $v = 0$ has the form

$$\phi = CJ_1(v) \equiv CJ_1(\gamma r) \tag{V.81}$$

in which case

$$w(r) = C\gamma J_0(\gamma r) + C_1 \tag{V.82}$$

with C_1 an arbitrary constant of integration.

If the plate is clamped along its edge at $r = a$, so that $w = \phi = 0$ for $r = a$, then, by virtue of (V.81), $J_1(\gamma a) = 0$; the smallest (approximate) root of this equation is $\gamma = 3.832/a$ in which case

$$(N_r)_{cr} = \frac{K}{a^2}(3.832)^2 \tag{V.83a}$$

so that

$$(N_T)_{cr} = -\frac{K(1-\nu)}{a^2}(3.832)^2 \tag{V.83b}$$

A second case considered in [24] is one for which the temperature distribution is such that

$$M_T^* = \frac{M_0}{1-\nu}\left(1 - \frac{v^2}{\gamma^2 a^2}\right) \tag{V.84}$$

In this case, (V.79) becomes

$$\frac{d^2\phi}{dv^2} + \frac{1}{v}\frac{d\phi}{dv} + \left(1 - \frac{1}{v^2}\right)\phi = -\frac{2M_0}{\gamma^3 a^2(1-\nu)}v \tag{V.85}$$

whose solution is given by

$$\phi = C_2 J_1(v) - \frac{2M_0}{\gamma^3 a^2(1-\nu)} v \tag{V.86}$$

From (V.86) it follows that

$$w = C_3 + \frac{M_0 r^2}{\gamma^2 a^2(1-\nu)} - \frac{C_2}{\gamma} J_0(\gamma r) \tag{V.87}$$

Imposing upon (V.86), (V.87) the conditions at $r = a$ for a clamped plate we are led to the solution

$$w(r) = \frac{2M_0[J_0(\gamma a) - J_0(\gamma r)]}{a^2\gamma^3(1-\nu)J_1(\gamma a)} - \frac{M_0}{\gamma^2(1-\nu)} + \frac{M_0 r^2}{\gamma^2 a^2(1-\nu)} \tag{V.88}$$

The deflection in (V.88) becomes infinite when $J_1(\gamma a) = 0$ which again leads to the critical values reflected in (V.83a,b); the absolute value of $(N_T)_{cr}$ in (V.83b) yields the critical buckling temperature.

In [8] a somewhat different presentation of buckling of an elastic, isotropic circular plate subjected to an axisymmetric temperature distribution is given; the temperature distribution is taken in the form

$$\delta T(r) = T_0(r) + zT_1(r) \tag{V.89}$$

which is, of course, a special case of (III.24). Thus

$$\begin{cases} N_T^* = \dfrac{E\alpha h}{1-\nu} T_0(r) \\[2ex] M_T^* = \dfrac{E\alpha h^3}{12(1-\nu)} T_1(r) \end{cases} \tag{V.90}$$

The radial and circumferential stress fields in this situation are given, respectively by

$$\begin{cases} \sigma_{rr} = \dfrac{E}{1-\nu^2}(e_{rr} + \nu e_{\theta\theta}) - \dfrac{E\alpha}{1-\nu}(T_0 + zT_1) \\ \sigma_{\theta\theta} = \dfrac{E}{1-\nu^2}(e_{\theta\theta} + \nu e_{rr}) - \dfrac{E\alpha}{1-\nu}(T_0 + zT_1) \end{cases} \tag{V.91}$$

We note that (III.19) follows from (V.90), (V.91) if (V.91) is integrated through the thickness of the plate. In (V.91), of course

$$\begin{cases} e_{rr} = v_{,r} + \dfrac{(w_{,r})^2}{2} - zw_{,rr} \\ e_{\theta\theta} = \dfrac{1}{r}v - \dfrac{z}{r}w_{,r} \end{cases} \tag{V.92}$$

where $v = v(r)$ again denotes the radial displacement of the plate. Combining (V.91) and (V.92) the resultant forces and moments are easily shown to have the form

$$\begin{cases} N_r = \dfrac{E}{1-\nu^2}\left[v_{,r} + \dfrac{1}{2}(w_{,r})^2 + \dfrac{\nu}{r}v - (1+\nu)\alpha T_0\right] \\ N_\theta = \dfrac{Eh}{1-\nu^2}\left[\dfrac{1}{r}v + \nu(v_{,r} + \dfrac{1}{2}(w_{,r}^2)) - (1+\nu)\alpha T_0\right] \end{cases} \tag{V.93}$$

and

$$\begin{aligned} M_r &= -K\left[w_{,rr} + \frac{\nu}{r}w_{,r} + (1+\nu)\alpha T_1\right] \\ M_\theta &= -K\left[\frac{1}{r}w_{,r} + \nu w_{,rr} + (1+\nu)\alpha T_1\right] \end{aligned} \tag{V.94}$$

In [8], Nowacki works directly with the equilibrium equations, for the symmetric situation, in the form

$$\begin{cases} (rN_r)_{,r} - N_\theta = 0 \\ (rM_{rr})_{,r} - M_\theta - Q_r = 0 \\ (rQ)_{,r} - (rN_r w_{,r})_{,r} - tr = 0 \end{cases} \tag{V.95}$$

with Q the shear force and t the external loading of the plate. By inserting Q from the second equation in (V.95) into the third equation in this set, and then employing the relations (V.94), one obtains

$$K\,\nabla_r^4\, w + K(1+\nu)\alpha\,\nabla_r^2\, T_1 = \frac{1}{r}(rN_r w_{,r})_{,r} + t \tag{V.96}$$

with $\nabla_r^4 = \nabla_r^2\nabla_r^2$ and $\nabla_r^2 = \partial_r^2 + \frac{1}{r}\partial_r$. Equation (V.96) is, of course, the form assumed by the first of the von Karman equations in (III.23) under the assumption of radial symmetry.

From the relations (V.93) one obtains the radial displacement $v(r)$ in the form

$$v(r) = \frac{r}{Eh}(N_\theta - \nu N_r) + \alpha r T_0 \tag{V.97}$$

or, if we use the first of the relations in (V.95) to eliminate the resultant stress N_θ in (V.97)
or, if we use the first of the relations in (V.95) to eliminate the resultant stress N_θ in (V.97)

$$v(r) = \frac{r}{Eh}\left[(rN_r)_{,r} - \nu N_r\right] + \alpha r T_0 \tag{V.98}$$

Finally, inserting (V.98) into the first of the relations in (V.93) we obtain

$$r\left[\frac{1}{r}(r^2 N_r)_{,r}\right]_{,r} + \frac{1}{2}Eh(w_{,r})^2 = 0 \tag{V.99}$$

Equation (V.99) is just the form assumed by the condition of geometric compatability of the strains for the special problem under consideration. To illustrate the utility of the relations (V.96), (V.99) with respect to axially symmetric temperature distributions of the form (V.89), Nowacki [8] presents, following Vinokurov [25], the solution of the clamped circular plate problem for which the boundary conditions are

$$\left\{\begin{array}{c} w(a) = 0, \; w_{,r}(a) = 0 \\ v(a) = \dfrac{a}{Eh}\left[(rN_r)_{,r} - \nu N_r\right]_{r=a} + \alpha a T_0 = 0 \end{array}\right. \tag{V.100}$$

At $r = 0$ it is required, as usual, that $r^{-1}w_{,r}$ and N_r be finite. Assuming that $T_1 =$ const. and $t =$ const. simplifies (V.96), after an integration, to the form

$$K\left(\frac{1}{r}(rw_{,r})_{,r}\right)_{,r} = N_r w_{,r} + \frac{tr}{2} \tag{V.101}$$

We now introduce the dimensionless variables

$$\left\{\begin{array}{l} \rho = \dfrac{r}{a}, \; \phi = w_{,r} = \dfrac{1}{a}w_{,\rho} \\ \psi = \dfrac{rN_r}{Eah}, \; k = 12(1-\nu^2)\dfrac{a^2}{h^2}, \; m = 6(1-\nu^2)\dfrac{ta^2}{Eh^3} \end{array}\right. \tag{V.102}$$

in terms of which (V.99) and (V.101) assume the respective forms

$$\left\{\begin{array}{l} \rho\psi_{,\rho\rho} + \psi_{,\rho} - \dfrac{1}{\rho}\psi = \dfrac{1}{2}\phi^2 \\ \\ \rho\phi_{,\rho\rho} + \phi_{,\rho} - \dfrac{1}{\rho}\phi = -k\phi\psi + m\rho^2 \end{array}\right. \tag{V.103}$$

while the boundary conditions (V.100) become

$$\phi(0) = 0,\ \phi(1) = 0,\ [\psi_{,\rho} - \nu\psi - \alpha T_0]_{\rho=1} = 0 \tag{V.104}$$

with $\frac{1}{\rho}\psi$ and v bounded at $\rho = 0$.

The system (V.103) may be solved (approximately) by applying Galerkin's method: as a first approximation we use the classical solution for the deflection of an isotropic, elastic, circular plate loaded by the uniform loading $t =$ const., namely,

$$\begin{cases} w^{(1)}(\rho) & = w_0[2(1-\rho^2) - (1-\rho^4)] \\ w_0 & = ta^4/64K \end{cases} \tag{V.105}$$

Then, as $\phi = \frac{1}{a}w_{,\rho}$

$$\phi^{(1)} = c(\rho^3 - \rho),\ c = 4w_0/a \tag{V.106}$$

Substituting $\phi^{(1)}$ for ϕ in the first equation in (V.103), and using the boundary conditions, yields

$$\begin{cases} \psi^{(1)} = \dfrac{c^2}{96}(\rho^7 - 4\rho^5 + 6\rho^3 - b\rho) \\ \\ b = \dfrac{5-3\nu}{1-\nu} - \dfrac{96}{c^2}\dfrac{\alpha T_0}{1-\nu} \end{cases} \tag{V.107}$$

If we now substitute $\psi^{(1)}$, from (V.107), for ψ in the second equation in (V.103), multiply this equation by $\rho^3 - \rho$, and integrate from $\rho = 0$ to $\rho = 1$, we obtain

$$c + c^3\frac{k}{64}\left(\frac{b}{24} - \frac{1}{14}\right) = \frac{1}{8}m \tag{V.108}$$

In (V.108) we recall that $c = \frac{4}{a}w_0$, while the presence of $b = \frac{5-3\nu}{1-\nu} - \frac{96}{c^2}\frac{\alpha T_0}{1-\nu}$ accounts for the influence of the temperature distribution on the plate deflection (which is seen to depend only on the function $T_0(r)$). For the special case in which $t = 0$ (so that $m = 0$ in (V.108)) it is possible to obtain from (V.108) the critical buckling temperature in the form

$$(T_0)_{cr} \simeq \frac{4}{3(1+\nu)} \cdot \frac{h^2}{\alpha a^2} \tag{V.109}$$

Note is made, in [8], of the fact that (V.109) is independent of the Young's modulus E of the plate.

For results on the thermoelastic stability of cylindrically orthotropic (laminated) circular plates the reader is referred to the paper of Stavsky [17] in which axisymmetric stability and thermal buckling equations are established for circular plates; the plates in [17] are composed of polar orthotropic layers subjected to mechanical loads depending only on the radial variable r and thermal fields of the form $\delta T = \delta T(r, z)$. For the initial buckling (eigenvalue) problems in [17], closed form solutions are presented in terms of Bessel functions of the first kind (and of fractional order) and the variation of the critical loads with the anisotropy parameters is analyzed. It is worth noting that, unlike the situation for isotropic circular plates, where (V.109) holds for the specific problem considered above (so that $(T_0)_{cr}$ is independent of E), for the same problem for a polar orthotropic circular plate Stavsky [17] obtains the result

$$(T_0)_{cr} \simeq \frac{4}{3} \cdot \frac{D_{rr}}{A_r h a^2} \tag{V.110}$$

where

$$\begin{cases} D_{rr} = \displaystyle\int_{-h/2}^{h/2} E_{rr} z dz \\ A_r = -(\alpha_r E_{rr} + \alpha_\theta E_{r\theta}) \end{cases} \tag{V.111}$$

VI. APPLICATIONS OF THE BERGER'S APPROXIMATION TO THE THERMAL BUCKLING AND POSTBUCKLING BEHAVIOR OF RECTANGULAR AND CIRCULAR PLATES

For the case of isotropic, linearly elastic response we have shown that the von Karman theory for large thermal deflections of a thin plate leads to the coupled system of nonlinear partial differential equations (III.4a,b), when rectangular Cartesian coordinates are used, and to the system (III.23) when polar coordinates are employed; these systems, together with associated boundary conditions can not be expected to yield exact solutions. Among the many approximate methods which have been devised to treat large deflection von Karman systems is the technique first promulgated by Berger in [26] for thin plates subjected to mechanical loading. As Tauchert [3] notes "the method is based on neglecting the strain energy associated with the second invariant of the strains in the plate middle surface. Although there does not appear to be any physical justification for this approximation, comparisons of the results with known solutions indicate that for a broad range of problems Berger's approach yields sufficiently accurate results." In fact all of the applications of Berger's approximation in literature, where accurate results appear to be obtained, are associated with problems in which the edges of the plate are constrained against in-plane movements; Nowinski and Ohnabe [27] have shown that, for the case of mechanical loading, the accuracy of Berger's method is closely tied to this specific type of boundary condition. In fact, by applying Galerkin's method to both the usual von Karman large deflection equations, and the approximate equations governing large deflections which are obtained from Berger's approach, it was determined in [27] that Berger's method is susceptible to serious errors if the edge (or edges) of the plate are free to move. It has been surmised in [3] that the conclusions reached in [27] are almost certainly extendible to the case of thermal deflections. The accuracy of Berger's method was also assessed by Jones, Mazumdar, and Cheung in [28] for the special case in which $t \equiv 0$, $M^T \equiv 0$ in (III.4a,b), and a perturbation technique was proposed for use in those situations where Berger's technique may not be sufficiently accurate. We will comment, below, on the work in [28] as well as on the work of Banerjee and Datta [29] who proposed a modified energy expression which leads, as in the case of Berger's method, to large deflection plate equations which are decoupled. Although only mechanical loadings are considered in [29], the results which are given for transverse loading of a circular plate are in good agreement with prior results for this problem for both movable and immovable edges and an extension of the method to the case of thermal loading has been indicated.

Berger's method was originally formulated, as we have already indicated, for isothermal plate problems; it was extended to non-isothermal problems by Basuli in [31] who used the technique to treat large deflection problems for rectangular and circular isotropic plates subjected to normal pressure and heating. We will begin our study, in this section, of the applications of Berger's method to large thermal deflection problems for thin plates by reviewing the work in [31] and the related work of Pal [32] which also assumes isotropic response. Then we will consider, in some detail, the application of Berger's technique to large thermal deflection problems for heated rectilinearly orthotropic rectangular plates as presented in Biswas [33]; this is followed by a brief description of the similar results for polar orthotropic circular plates which were described by Pal [34].

The starting point for the application of Berger's approximation to large deflection problems for isotropic rectangular plates is the observation that the total potential energy of the plate, which is given by (III.41), may be rewritten in the form

$$\begin{aligned}\Pi = \int_0^b \int_0^a \Bigg\{ & \frac{Eh}{2(1-\nu^2)}[\epsilon_1 - 2(1-\nu)\epsilon_2] \\ & + \frac{1}{2}K\left[(\nabla^2 w)^2 - 2(1-\nu)\left\{(w_{,xy}^2 - w_{,xx}w_{,yy})\right\}\right] \\ & - N^T\epsilon_1 + M^T \nabla^2 w - tw \Bigg\} dxdy\end{aligned} \tag{VI.1}$$

where ϵ_1 and ϵ_2 are, respectively, the first and second strain invariants of the middle surface of the plate, i.e.,

$$\epsilon_1 = \epsilon_{xx}^0 + \epsilon_{yy}^0 = u_{,x} + v_{,y} + \frac{1}{2}\left(w_{,x}^2 + w_{,y}^2\right) \tag{VI.2a}$$

$$\begin{aligned}\epsilon_2 = \epsilon_{xx}^0\epsilon_{yy}^0 - \frac{1}{4}(\gamma_{xy}^0)^2 & \\ = u_{,x}v_{,y} + \frac{1}{2}u_{,x}w_{,y}^2 + \frac{1}{2}v_{,y}w_{,x}^2 & \\ -\frac{1}{4}\left\{u_{,y}^2 + v_{,x}^2 + 2u_{,y}v_{,x} + 2u_{,y}w_{,x}w_{,y} + 2v_{,x}w_{,x}w_{,y}\right\} &\end{aligned} \tag{VI.2b}$$

Assuming a temperature distribution δT of the form

$$\delta T(x,y,z) = T_0(x,y) + g(z)T_1(x,y) \tag{VI.3}$$

Basuli [31] writes (VI.1) in the form

$$\begin{aligned}\Pi = \int_0^b \int_0^a \Bigg[\frac{1}{2} K \Bigg\{ & (\nabla w)^2 + \frac{12}{h^2}\epsilon_1^2 \\ & -2(1-\nu)\left[\frac{12}{h^2}\epsilon_2 + w_{,xx}w_{,yy} - w_{,xy}^2\right] \Bigg\} - tw \\ & -\frac{E\alpha}{1-\nu}\left\{\epsilon_1 h T_0 - f(h) T_1 \nabla^2 w\right\} \Bigg] dxdy\end{aligned} \tag{VI.4}$$

where

$$f(h) = \int_{-h/2}^{h/2} g(z) z dz$$

The functional Π is now minimized using the Euler-Lagrange variational equations

$$\frac{\partial \Pi}{\partial u} - \frac{\partial}{\partial x}\frac{\partial \Pi}{\partial u_{,x}} - \frac{\partial}{\partial y}\frac{\partial \Pi}{\partial u_{,y}} = 0 \tag{VI.5a}$$

$$\frac{\partial \Pi}{\partial v} - \frac{\partial}{\partial x}\frac{\partial \Pi}{\partial v_{,x}} - \frac{\partial}{\partial y}\frac{\partial \Pi}{\partial v_{,y}} = 0 \tag{VI.5b}$$

$$\begin{aligned}\frac{\partial \Pi}{\partial w} - \frac{\partial}{\partial x}\frac{\partial \Pi}{\partial w_{,x}} - \frac{\partial}{\partial y}\frac{\partial \Pi}{\partial w_{,y}} + \frac{\partial^2}{\partial x^2}\frac{\partial \Pi}{\partial w_{,xx}} \\ + \frac{\partial^2}{\partial x \partial y}\frac{\partial \Pi}{\partial w_{,xy}} + \frac{\partial^2}{\partial y^2}\frac{\partial \Pi}{\partial w_{,yy}} = 0\end{aligned} \tag{VI.5c}$$

which yields (upon following Berger [26] and setting $\epsilon_2 = 0$) the relations

$$\frac{\partial}{\partial x}\left(\frac{Eh}{1-\nu^2}\epsilon_1 - N^T\right) = 0, \ \frac{\partial}{\partial y}\left(\frac{Eh}{1-\nu^2}\epsilon_1 - N^T\right) = 0 \tag{VI.6a}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial x^2}(K \nabla^2 w + M^T) + \frac{\partial^2}{\partial y^2}(K \nabla^2 w + M^T) \\ -\frac{\partial}{\partial x}\left\{\left(\frac{Eh}{1-\nu^2}\epsilon_1 - N^T\right) w_{,x}\right\} \\ -\frac{\partial}{\partial y}\left\{\left(\frac{Eh}{1-\nu^2}\epsilon_1 - N^T\right) w_{,y}\right\} - t = 0\end{aligned} \tag{VI.6b}$$

in the general case and

$$\frac{\partial}{\partial x}\left\{\epsilon_1 - (1+\nu)\alpha T_0\right\} = 0, \ \frac{\partial}{\partial y}\left\{\epsilon_1 - (1+\nu)\alpha T_0\right\} = 0 \tag{VI.7a}$$

$$
\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left\{K \nabla^2 w + \frac{E\alpha}{1-\nu} f(h) T_1\right\} + \frac{\partial^2}{\partial y^2}\left\{K \nabla^2 w + \frac{E\alpha}{1-\nu} f(h) T_1\right\} \\
&\qquad - \frac{E}{1-\nu}\frac{\partial}{\partial x}\left\{ [\epsilon_1 - (1+\nu)\alpha T_0]\, w_{,x} \right\} \\
&\qquad\qquad - \frac{\partial}{\partial y}\left\{ \left(\frac{Eh}{1-\nu^2}\epsilon_1 - N^T\right) w_{,y} \right\} - t = 0
\end{aligned}
\tag{VI.6b}
$$

in the general case and

$$
\frac{\partial}{\partial x}\left\{\epsilon_1 - (1+\nu)\alpha T_0\right\} = 0,\ \frac{\partial}{\partial y}\left\{\epsilon_1 - (1+\nu)\alpha T_0\right\} = 0 \tag{VI.7a}
$$

$$
\begin{aligned}
&\frac{\partial^2}{\partial x^2}\left\{K \nabla^2 w + \frac{E\alpha}{1-\nu} f(h) T_1\right\} + \frac{\partial^2}{\partial y^2}\left\{K \nabla^2 w + \frac{E\alpha}{1-\nu} f(h) T_1\right\} \\
&\qquad - \frac{E}{1-\nu}\frac{\partial}{\partial x}\left\{ [\epsilon_1 - (1+\nu)\alpha T_0] w_{,x} \right\} \\
&\qquad\qquad - \frac{E}{1-\nu}\frac{\partial}{\partial y}\left\{ [\epsilon_1 - (1+\nu)\alpha T_0] w_{,y} \right\} - t = 0
\end{aligned}
\tag{VI.7b}
$$

in the special case in which $\delta T(x,y,z)$ is given by (VI.3).

From (VI.6a) one easily obtains

$$
\frac{Eh}{1-\nu^2}\epsilon_1 - N^T = \lambda^2,\ (\lambda^2 \text{ const.}) \tag{VI.8}
$$

which upon substitution into (VI.6b) yields

$$
K \nabla^2 (\nabla^2 - \lambda^2) w = t - \nabla^2 M^T \tag{VI.9}
$$

As Tauchert [3] has noted, the system consisting of (VI.8), (VI.9) is much simpler than the corresponding Von Karman system (III.4a,b); the equations (VI.8), (VI.9) are quasi-linear and decoupled in the following sense: (VI.9) is linear in w and may be solved independently of (VI.8). Equation (VI.8) is linear in $u_{,x}$ and $v_{,y}$ and may be solved once w has been determined. The only coupling that remains in (VI.8), (VI.9) is through the parameter λ. In a similar manner, for the special case governed by (VI.3), (VI.7a) yields

$$
\epsilon_1 - (1+\nu)\alpha T_0 = \frac{\mu^2 h^2}{12},\ (\mu^2 \text{ const.}) \tag{VI.10}
$$

whose substitution into (VI.7b) produces

$$K \nabla^2 (\nabla^2 - \mu^2) w = t - \frac{\alpha E}{1-\nu} f(h) \nabla^2 T_1 \quad . \tag{VI.11}$$

We now consider the application of the system (VI.10), (VI.11) to the thermal buckling of a simply supported rectangular plate which occupies the domain $0 \le x \le a$, $0 \le y \le b$ in the x, y plane; in this case the assumed temperature distribution has the form given by (VI.3).

Remarks: In [31] Basuli writes the conditions of simple support along the edges of the plate in the form

$$\begin{cases} w = w_{,xx} = 0, \;\; x = 0, a; \;\; 0 \le y \le b \\ w = w_{,yy} = 0, \;\; y = 0, b; \;\; 0 \le x \le a \end{cases} \tag{VI.12}$$

However, for the simply supported plate

$$\begin{aligned} K(w_{,xx} + \nu w_{,yy}) + M^T &= 0, \;\; x = 0, a; \;\; 0 \le y \le b \\ K(w_{,yy} + \nu w_{,xx}) + M^T &= 0, \;\; y = 0, b; \;\; 0 \le x \le a \end{aligned}$$

with $w = 0$ along each of the four edges; thus (VI.12) holds, in the case of simply supported edges if and only if $M^T(0, y) = M^T(a, y) = 0$, for $0 \le y \le b$, and $M^T(x, 0) = M^T(x, b) = 0$, for $0 \le x \le a$. For the temperature distribution represented by (VI.3)

$$M^T = \frac{\alpha E}{1-\nu} T_1(x, y) \int_{-h/2}^{h/2} g(z) z dz \equiv \frac{\alpha E}{1-\nu} f(h) T_1(x, y) \tag{VI.13}$$

and, thus, (VI.12) represents the conditions of simple support along the edges of the plate if and only if

$$\begin{cases} T_1(0, y) = T_1(a, y) = 0; \;\; 0 \le y \le b \\ T_1(x, 0) = T_1(x, b) = 0; \;\; 0 \le x \le a \end{cases} \tag{VI.14}$$

In order to maintain the validity of the analysis in [31] we will assume, below, that the function T_1 satisfies (VI.14).

Proceeding with the analysis in [31] we assume that each of $t(x, y)$, $T_1(x, y)$ may be expanded in a double Fourier series in the products $\sin \dfrac{m\pi x}{a} \sin \dfrac{n\pi y}{b}$ so that, in particular, on the right-hand side of (VI.11)

$$t - \frac{\alpha E}{1-\nu} f(h) \nabla^2 T_1 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} q_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{VI.15}$$

Also, in view of (VI.12), $w(x, y)$ is sought in the form

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{VI.16}$$

Assuming the edges of the plate are constrained against in-plane motion is equivalent to requiring that

$$\begin{cases} u(0, y) = u(a, y) = 0; \; 0 \leq y \leq b \\ v(x, 0) = v(x, b) = 0; \; 0 \leq x \leq a \end{cases} \tag{VI.17}$$

By substituting (VI.15) and (VI.16) into (VI.11) we obtain, in the usual manner, the relations

$$w_{mn} = \frac{q_{mn}}{K\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]\left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \mu^2\right]} \tag{VI.18}$$

so that the deflection has been determined up to the constant μ^2. We now write out (VI.10) in the form

$$u_{,x} + v_{,y} + \frac{1}{2} w_{,x}^2 + \frac{1}{2} w_{,y}^2 - (1+\nu)\alpha T_0 = \frac{\mu^2 h^2}{12} \tag{VI.19}$$

Integration of (VI.19) over the domain of the plate, and use of the boundary conditions (VI.17), which express the constraint of the plate against in-plane motion, yields the condition

$$\int_0^a \int_0^b \left\{ \frac{1}{2} \left(w_{,x}^2 + w_{,y}^2 \right) - (1+\nu)\alpha T_0 \right\} dxdy = \frac{\mu^2 h^2}{12} ab \tag{VI.20}$$

We now substitute the expansion (VI.16) into (VI.20) and use standard results involving trigonometric integrals, e.g.,

$$\int_0^a \int_0^b \cos^2 \frac{m\pi x}{b} \sin^2 \frac{n\pi y}{b} dxdy = \frac{1}{4} ab$$

so as to obtain from (VI.20) the equation

$$\begin{aligned}\frac{ab}{8}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} w_{mn}^2\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\\ -(1+\nu)\alpha\int_0^a\int_0^b T_0(x,y)dxdy=\frac{\mu^2h^2}{12}ab\end{aligned} \tag{VI.21}$$

The further substitution of the expressions for the w_{mn} in (VI.18) into (VI.21) then produces an algebraic relation from which μ^2 may be determined; once μ^2 has been computed (VI.16), (VI.18) serve to completely determine the deflection $w(x,y)$.

Remarks: Basuli [31] considers a specific example for which

$$T_0(x,y)=T_0=const.,\ T_1(x,y)=T_1\cos\left(\frac{2\pi y}{b}\right) \tag{VI.22}$$

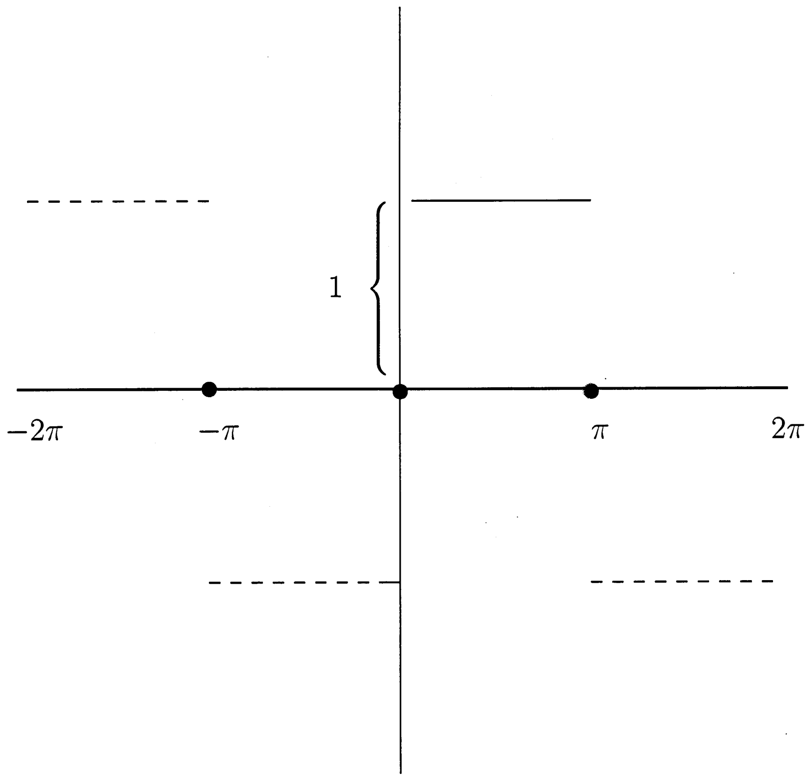
however, the subsequent results are specious in as much as $T_1(x,y)$ does not satisfy (VI.14). The source of confusion in much of the literature maybe traced to the following elementary fact: any piecewise continuous function on the domain $0\le x\le a$, $0\le y\le b$, say, $M^T(x,y)$, can be extended to the entire x,y plane in such a manner that the extended function may be represented by the Fourier series

$$M^T(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{VI.23}$$

However, while the series on the right hand side of (VI.23) converges to the value of $M^T(x,y)$ at each point of continuity of M^T in the domain $0<x<a$, $0<y<b$, it will not converge to the values of M^T along the boundary of the domain; it is the function $M^T(x,y)$, as defined by the temperature distribution $\delta T(x,y)$, and not its Fourier series representation, which must satisfy the conditions (VI.14) in order that the criteria for a simply supported rectangular plate, under thermal loading, reduce to (VI.12). In a one-dimensional setting this phenomena is easily represented by the Fourier sine series expansion of $f(x)=1$ on $(0,\pi)$, i.e.

$$1=\frac{4}{\pi}\sum_{k=1}^{\infty}\sin(2k-1)x,\ 0<x<\pi \tag{VI.24}$$

For all x, $0 < x < 1$. the series on the right-hand side of (VI.24) converges to 1; however, convergence breaks down at $x = 0$, $x = \pi$ where the series converges, by the Fourier theorem, to the average value of the left and right hand limits (of the periodic extension, with period 2π, of the odd extension of $f(x)$ to $(-\pi, \pi)$). The situation is depicted below:



The second application of Berger's method that we want to consider is to the problem of axisymmetric deformations of a solid circular elastic plate subjected to a uniform load t and a stationary temperature field $\delta T(r, z)$. In this case equations (VI.8), (VI.9) assume the form

$$\frac{Eh}{1-\nu^2}\left[u_{,r} + \frac{1}{r}u + \frac{1}{2}w_{,r}^2\right] - N^T = K\lambda^2 \tag{VI.25}$$

$$K\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \lambda^2\right)w = t - \nabla^2 M^T \tag{VI.26}$$

Under the assumption that $t' = t - \nabla^2 M^T$ is a constant, Tauchert [3] obtains a solution to (VI.26) in the form

$$w = C_1 I_0(\lambda r) + C_2 - \frac{t' r^2}{4K\lambda^2} \tag{VI.27}$$

with Φ_0 the modified Bessel function of the first kind of order zero and C_1, C_2 arbitrary constants. If the edge of the plate at $r = b$ is constrained against in-plane motion so that $u(b) = 0$ then integration of (VI.25) over the domain $0 \leq r \leq b$, $0 \leq \theta < 2\pi$, yields the relation

$$\begin{aligned} &\frac{1}{2}C_1^2 \left\{ \frac{1}{2}\lambda^2 b^2 \left[I_1^2(\lambda b) - I_0^2(\lambda b) \right] + \lambda b I_1(\lambda b) I_0(\lambda b) \right\} \\ &- \frac{C_1 t' b^2 I_2(\lambda b)}{K\lambda^2} + \frac{t' b^4}{16K^2\lambda^4} - \frac{1-\nu^2}{Eh}\int_0^b N^T r dr = b^2 t^2 \lambda^2 / 24 \end{aligned} \tag{VI.28}$$

Assuming the plate to be clamped, so that $w(b) = \dfrac{dw}{dr}(b) = 0$, we obtain from (VI.27)

$$w = -\frac{t' b^2}{4K\lambda^2}\left\{ \frac{2[I_0(\lambda b) - I_0(\lambda r)]}{\lambda b I_1(\lambda b)} - 1 + \frac{r^2}{b^2} \right\} \tag{VI.29}$$

while the relation (VI.28) for λ assumes the form

$$\left(\frac{t' b^4}{Kt}\right)^2 = \frac{\left\{ \frac{1}{3}(\lambda b)^6 + \dfrac{8\displaystyle\int_0^b N^T r dr}{\left(\dfrac{Eh}{1-\nu^2}\right) t^2}(\lambda b)^4 \right\}}{\left\{ 1 + \dfrac{I_0(\lambda b) - 4I_2(\lambda b)}{\lambda b I_1(\lambda b)} - \dfrac{I_0^2(\lambda b)}{2I_1^2(\lambda b)} \right\}} \tag{VI.30}$$

With the same assumption of circular symmetry, and a temperature distribution of the form given by

$$\delta T(r, z) = T_0(r) + g(z) T_1(r).$$

Basuli [31] writes the system (VI.25), (VI.26) in the form

$$\frac{du}{dr} + \frac{v}{r} + \frac{1}{2}w_{,r}^2 - (1+\nu)\alpha T_0 = \frac{\beta^2 h^2}{12} \tag{VI.31}$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{2}\frac{d}{dr}\right)\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\beta^2\right)w=\frac{1}{K}\left[t-\frac{\alpha E}{1-\nu}f(h)\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)T_1\right] \tag{VI.32}$$

and obtains, for a uniform load distribution t, that special case of (VI.27) which has the form

$$w=AI_0(\beta r)+B-\frac{t'}{4K\beta^2}r^2 \tag{VI.33}$$

with $t'=t-\frac{\alpha E}{1-\nu}f(h)\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right)T_1$. Clearly, $t'=t$ if either $f(h)=0$ or if T_1 is harmonic. With $u(b)=0$, and $w(b)=\frac{dw}{dr}(b)=0$, the solution of the system (VI.31), (VI.32) follows the same pattern as that outlined above for temperature distributions more general than (VI.3) and will, therefore, not be repeated here. Somewhat more problematic, however, is the solution of (VI.31), (VI.32) which is presented by Basuli in [31] for the case of a simply supported circular plate subject to the temperature distribution (VI.3) and the assumption of constrained in-plane motion; in [31] Basuli takes the conditions of simple-support along the edge of the plate at $r=b$ to have the form $w=0$, at $r=b$, and

$$\frac{d^2w}{dr^2}+\frac{w}{r}\frac{dw}{dr}=0,\quad \text{at } r=b \tag{VI.34}$$

However, by virtue of (III.26), with $w_{,\theta}=0$, and $R_1=0,\ R_2=b$, we have, along the edge at $r=b$, $w(b)=0$ and

$$K\left(\frac{d^2w}{dr^2}+\frac{1}{r}\frac{dw}{dr}\right)\bigg|_{r=b}+M_T^*|_{r=b}=0 \tag{VI.35}$$

The problem that arises now is the same one which has been discussed, at length, earlier in this section, namely, in view of (III.22)

$$M_T^*|_{r=b}=\frac{E\alpha}{1-\nu}\int_{-h/2}^{h/2}\delta T(b,z)z\,dz=\frac{E\alpha}{1-\nu}f(h)T_1(b) \tag{VI.36}$$

so that (VI.34) follows from (VI.36) if and only if $f(h)=0$ or $T_1(b)=0$; under this latter assumption, the results in [31] are valid and the full solution of the boundary-value problem associated with (VI.31), (VI.32), when the edge at $r=b$ is simply supported, assumes the form

$$w=\frac{-t'b^4}{4K\beta^3b^3H}\left\{2(1+\nu)[I_0(\beta b)-I_0(\beta r)]-H\beta b\left(1-\left(\frac{r}{b}\right)^2\right)\right\} \tag{VI.37}$$

$$\left(\frac{t'b^4}{Kh}\right)^2 = \frac{\frac{1}{3}(\beta b)^6 + \frac{(1+\nu)\alpha F(b)}{h^2}\beta^4 b^4}{\frac{1}{2} + \left\{\frac{1}{2}[I_1^2(\beta b) - I_0^2(\beta b)] + \frac{I_0(\beta b)I_1(\beta b)}{\beta b}\right\} - \frac{4(1+\nu)I_2(\beta b)}{H\beta b}} \tag{VI.38}$$

where

$$\begin{cases} H = \beta b I_0(\beta b) - (1-\nu)I_1(\beta b) \\ F(b) = \int_0^b T_0(r)rdr \end{cases} \tag{VI.39}$$

Although Basuli's work [31], extending Berger's method [26] to non-isothermal plate deflection problems, preceded the work of Pal [34], the latter author appears not to have been aware of the earlier research in this direction. Thus, Pal [34] rederives the Berger's type equations by minimizing the total potential energy associated with a circular plate under the assumption of a radially symmetric temperature, i.e., equations (2.9), (2.10a) of [34] are completely equivalent to (VI.31), (VI.32) (which appear in [31]); these decoupled large deflection equations are then solved, in [34], for the case in which $t \equiv 0$, by using the method of successive approximations. Included in the analysis in [34] are results for annular plates, i.e., circular plates with a concentric circular hole. In line with our related discussions in this section, the applicability of the results in [34] for circular and annular plates with simply supported edges is subject to the same constraints on the behavior of $T_1(r)$, along the plate edges, that have been delineated previously; in Pal [34], the temperature distribution is taken in the form

$$\delta T(\zeta, z) = \left\{\bar{T}_0 + \bar{T}_1(1-\zeta^2)\right\}\left(1 + \frac{2z}{3h}\right) \tag{VI.40}$$

where $\zeta = r/a$. Thus, for $\delta T(r,z)$ we have

$$\delta T = T_0(r) + g(z)T_1(r)$$

with

$$\begin{cases} T_0(r) = \bar{T}_0 + \bar{T}_1\left(1 - \left(\frac{r}{a}\right)^2\right) = T_1(r) \\ g(z) = \frac{2}{3}\cdot\frac{z}{h} \end{cases} \tag{VI.41}$$

It is clear, from (VI.40), that $T_1(a) = 0$ if and only if $\bar{T}_0 = 0$; in [34] the outer edge of an annular plate is located at $r = a$ while the inner edge is at $r = b$ (with $b = 0$ for a solid circular plate). Therefore, in Pal's work [34], (IV.34) is equivalent to (VI.35), with b replaced by a provided $\bar{T}_0 = 0$. In Figs. 23 and 24, which are taken from [34], we show the relation between the temperature rise and the normalized maximal deflection $\delta/h(\delta = w(0))$ at the center of the plate for the simply supported and clamped edge conditions, respectively; in these figures

$$\begin{cases} \tilde{T} = \int_{-h/2}^{h/2} \delta T(r,z)dz \\ \bar{T} = \int_{-h/2}^{h/2} \delta T(r,z)zdz \end{cases} \tag{VI.42}$$

Fig. 25 depicts, for an annular plate with aspect ratio $\frac{b}{a} = 0.4$, the relation between the temperature rise and the normalized deflection δ/h, for the same temperature distribution (VI.40); the plate in Fig. 25 has both inner and outer edges simply supported. For the same plate, and the same temperature distribution, the analogous results for an annular plate with both inner and outer edges clamped are depicted in Fig. 26. The results in [34] for the case where the same annual plate has its' inner edge clamped and its outer edge simply supported is shown in Fig. 27. The results shown in Fig. 25 (the annular plate with both edges simply supported) are open to the criticism that even if $\bar{T}_0 = 0$, so that $T_1(a) = 0$ is a direct consequence of (VI.41), i.e., it does not necessarily follow that $T_1(b) = 0$; in fact $T_1(b) = 0$, when $\bar{T}_0 = 0$, if and only if $a = b$ which reduces the problem to one for a solid circular plate of radius a.

Both Biswas [35] and Pal [36] have applied Berger's approximation method to linearly elastic thin plates exhibiting anisotropic response. In [35], Biswas solves the large-deflection problem for a rectilinearly orthotropic rectangular plate which is simply supported and subjected to a stationary temperature distribution while in [36] Pal obtains solutions for cylindrically orthotropic circular plates using the method of successive approximations; it is also demonstrated in [36] that the solutions for a polar orthotropic circular plate reduce to those for an isotropic circular plate as a special case. We now, briefly, review each of the two papers [35], [36] beginning with the work of Biswas [35] which is, unfortunately, subject, once again, to the same criticism concerning the interpretation of simple support boundary conditions in thermal deflection problems. The notation employed by Biswas in [35] parallels that of the earlier paper [22]; thus, in particular, (V.51) applies, i.e.

$$D_x = D_{11},\ H = D_1 + 2D_{66},\ D_y = D_{22} \tag{VI.43}$$

with $D_1 = \frac{1}{2}(D_{12}+D_{21})$, while the $\tilde{\beta}_i,\ i = 1, 2$, are given by (V.52). The first strain invariant ϵ_1 is given in this situation by

$$\epsilon_1 = \epsilon^0_{xx} + \kappa\epsilon^0_{yy},\ \kappa = \sqrt{\frac{D_y}{D_x}} \tag{VI.44}$$

and M_T, as in (V.50), is given by $M_T = \int_{-h/2}^{h/2} \delta T z dz$. The full set of von Karman large deflection equations has the form (V.50), (V.53) where the $E_1, E_2, \nu_{12}, \nu_{21}, G_{12}$, and the resultant (thermal) stresses $N_T,\ \tilde{N}_T^{\prime}, \tilde{N}_T^2$ have been defined in §V. Neglecting the second strain invariant ϵ_2, Biswas [35] writes the total potential energy for the rectilinearly orthotropic rectangular plate in the form

$$\begin{aligned}\Pi = \frac{1}{2}\int_0^b\int_0^a &\Big\{D_x w^2_{,xx} + 2D_1 w_{,xx}w_{,yy} \\ &+D_y w^2_{,yy} + 4D_{66}w^2_{,xy} + D_x\frac{12}{h^2}\epsilon_1^2\Big\}dxdy \\ + \int_{-h/2}^{h/2}\int_0^b\int_0^a &\Big\{\tilde{\beta}_1\epsilon_{xx}\delta T(x,y,z) + \tilde{\beta}_2\epsilon_{yy}\delta T(x,y,z)\Big\}\,dxdydz\end{aligned} \tag{VI.45}$$

The temperature variation in [35] is chosen so as to have the form

$$\delta T(x,y,z) = 2\theta\left(\frac{z}{h}\right),\ -\frac{h}{2} \le z \le \frac{h}{2} \tag{VI.46}$$

in which case

$$T\left(x,y,\frac{h}{2}\right) = \theta,\ T\left(x,y,-\frac{h}{2}\right) = -\theta \tag{VI.47}$$

Combining (VI.45) with (VI.46), and using Euler's equations (VI.5a,b,c), one obtains

$$\frac{\partial \epsilon_1}{\partial x} = 0, \; \frac{\partial \epsilon_1}{\partial y} = 0 \tag{VI.48a}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial x^2}&\left\{D_x w_{,xx} + D_1 w_{,yy} + \tilde{\beta}_1 M_T\right\} \\ &+\frac{\partial}{\partial y^2}\left\{D_y w_{,yy} + D_1 w_{,xx} + \tilde{\beta}_2 M_T\right\} \\ &\quad +4\frac{\partial^2}{\partial x^2}(D_{66} w_{,xy}) - \frac{\partial}{\partial x}\left\{D_x \frac{12}{h^2}\epsilon_1 w_{,x}\right\} \\ &\quad -\frac{\partial}{\partial y}\left\{D_y \frac{12}{h^2}\epsilon_1 \kappa w_{,y}\right\} = 0\end{aligned} \tag{VI.48b}$$

Clearly, (VI.48a) implies that

$$\epsilon_1 = \text{ const. } \equiv \frac{\alpha^2 h^2}{12} \tag{VI.49}$$

while (VI.48b), in conjunction with (VI.49), yields

$$\begin{aligned}D_x w_{,xxxx} &+ 2H w_{,xxyy} + D_y w_{,yy} \\ &-D_x \alpha^2 (w_{,xx} + \kappa w_{,yy}) + \tilde{\beta}_1 M_{T,xx} + \tilde{\beta}_2 M_{T,yy} = 0\end{aligned} \tag{VI.50}$$

For the temperature variation given by (VI.46) we have

$$M_T \equiv \int_{-h/2}^{h/2} \delta T(x,y,z) dz = \frac{1}{6}\theta h^2 = \text{ const.}$$

which Biswas [35] expands in the double Fourier series

$$M_T = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3\ldots}^{\infty} \frac{16 M_T}{mn\pi^2} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{VI.51}$$

While (VI.51) is correct as a Fourier representation of M_T, convergence of the series on the right-hand side of (VI.51) to M_T holds only on the open domain $0 < x < a$, $0 < y < b$; in particular, M_T does not vanish along the edges of the plate for the temperature variation chosen. For this reason, $\tilde{M}_T^1$ and $\tilde{M}_T^2$, as given by (III.13), do not vanish along the edges of the plate; therefore, satisfaction of the conditions expressing the fact that the plate is simply supported, i.e. (III.15), does not follow as a consequence of choosing a trial deflection

$w(x, y)$ with the property that both $w_{,xx}$ and $w_{,yy}$ vanish along the edges at $x = 0, a$, for $0 \leq y \leq b$, and along the edges $y = 0, b$, for $0 \leq x \leq a$ In spite of this, Biswas [35] chooses a trial deflection function of the form

$$w = \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{VI.52}$$

so that substitution of (VI.51) and (VI.52) into (VI.50) yields

$$A_{mn} = \frac{\dfrac{16M_T}{mn\pi^2}\left(\tilde{\beta}_1 \dfrac{m^2\pi^2}{a^2} + \tilde{\beta}_2 \dfrac{n^2\pi^2}{b^2}\right)}{D_x \dfrac{m^4\pi^4}{a^4} + 2H\dfrac{m^2n^2\pi^2}{a^2b^2} + D_y \dfrac{n^4\pi^4}{b^4} + D_x\alpha^2\left(\dfrac{m^2\pi^2}{a^2} + \dfrac{n^2\pi^2}{b^2}\sqrt{\dfrac{D_y}{D_x}}\right)} \tag{VI.53}$$

An additional criticism of the procedure followed in [35] would focus on the fact that as M_T is constant on the open domain $0 < x < a$, $0 < y < b$, both $M_{T,xx}$ and $M_{T,yy}$ vanish in (VI.50).

The relation (VI.53) for the coefficients A_{mn} still contains the unknown constant α^2 which arises in (VI.49). Assuming that

$$\begin{cases} u = 0, \text{ on } x = 0, a, \text{ for } 0 \leq y \leq b \\ v = 0, \text{ on } y = 0, b, \text{ for } 0 \leq x \leq a \end{cases} \tag{VI.54}$$

Biswas [35] takes the displacements u, v to have the form

$$\begin{cases} u = \displaystyle\sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \chi_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \\ v = \displaystyle\sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \chi_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \end{cases} \tag{VI.55}$$

in which case substitution of (VI.52) and (VI.55) into (VI.49), and integration over the domain $0 \leq x \leq a$, $0 \leq y \leq b$, yields

$$\sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{8} A_{mn}^2 \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right) = \frac{\alpha^2 h^2}{12} \tag{VI.56}$$

Substitution of (VI.53) into (VI.56) now serves to determine α^2 and, hence, $w(x, y)$; for the data

$$\frac{D_y}{D_x} = 0.32, \ \frac{H}{D_x} = 0.2, \ \frac{a}{h} = 20, \ \frac{\tilde{\beta}_2}{\tilde{\beta}_1} = 0.5$$

Fig. 28 depicts the predictions, for the theory delineated above, of the variation of the central deflection of the plate with changes in the temperature parameter $\theta \left(\frac{a}{h}\right)^2 \frac{\tilde{\beta}_1 h^3}{D_x}$.

The counterpart of [35], for the case of heated cylindrically orthotropic circular plates, is the work of Pal [36] who treats both dynamic as well as static behavior; Pal [36], as in [34], again uses, as an example, the temperature variation (VI.40). In [36], however, the correct form of the simply supported boundary condition is used and the resultant problem is treated by employing a series expansion of w in the variable $\zeta = \frac{r}{a}$. For the case of polar orthotropic response the total potential energy in the static case assumes the form

$$\begin{aligned}\Pi = &\frac{E_r h}{2(1-\nu_r\nu_\theta)} \int_0^{2\pi}\int_0^a [\tilde{e}_1^2 - 2(k-\nu_\theta)\tilde{e}_2] r\,dr\,d\theta \\ &+\frac{h^2}{12}\int_0^{2\pi}\int_0^a \left\{ \left[w_{,rr}^2 + \left(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta}\right)\right]^2 \right. \\ &-2(k-\nu_\theta)\left[w_{,rr}\left(\frac{1}{r}w_{,r} + \frac{1}{r^2}w_{,\theta\theta}\right)\right. \\ &\left.\left.-\frac{2G_{r\theta}(1-\nu_r\nu_\theta)}{E_r(k-\nu_\theta)}\left(\frac{1}{r}w_{,r\theta} - \frac{1}{r^2}w_{,\theta}\right)^2\right]\right\} r\,dr\,d\theta \\ &-\frac{2}{h}\left\{\int_0^{2\pi}\int_0^a \left[(\alpha_r + \nu_\theta\alpha_\theta)e_{rr}^0 + \frac{\nu_\theta}{\nu_r}(\alpha_\theta + \nu_r\alpha_r)\epsilon_{\theta\theta}^0\right]\tilde{T} r\,dr\,d\theta\right\}\end{aligned} \tag{VI.57}$$

with

$$\begin{cases} \tilde{e}_1 = e_{rr}^0 + k e_{\theta\theta}^0 \\ \tilde{e}_2 = e_{rr}^0 e_{\theta\theta}^0 - \dfrac{G_{r\theta}(1-\nu_r\nu_\theta)}{2E_r(k-\nu_\theta)}(e_{r\theta}^0)^2 \\ k = (\nu_\theta/\nu_r)^{1/2} \end{cases} \tag{VI.58a}$$

and

$$\bar{T} = \int_{-h/2}^{h/2} \delta T(r,z)\,dz, \ \tilde{T} = \int_{-h/2}^{h/2} \delta T(r,z)\,z\,dz \tag{VI.58b}$$

The functions $\tilde{e}_1$ and $\tilde{e}_2$ are, respectively, the first and second strain invariants of the middle surface of the polar orthotropic plate. Introducing the dimensionless variables

$$\tilde{w} = \frac{w}{h},\ \zeta = \frac{r}{a},\ T^* = \frac{\bar{T}}{h},\ \hat{T} = \frac{\tilde{T}}{h^2} \tag{VI.59}$$

and deleting the second strain invariant $\tilde{e}_2$ from (VI.57), the Euler-Langrange equations associated with (VI.57) assume the form

$$\begin{aligned}
&\frac{1}{\zeta}\frac{d}{d\zeta}\left\{\zeta\frac{d}{d\zeta}\left[\frac{1}{\zeta}\left(\zeta\frac{d\tilde{w}}{d\zeta}\right)\right]\right\} \\
&\quad = \frac{k^2-1}{\zeta^2}\left(\frac{d^2\tilde{w}}{d\zeta^2} - \frac{1}{\zeta}\frac{d\tilde{w}}{d\zeta}\right) \\
&\quad +12\left(\frac{a}{h}\right)^2\left\{\frac{1}{\zeta}\frac{d}{d\zeta}\left(\tilde{e}_1\zeta\frac{d\tilde{w}}{d\zeta}\right)\right. \\
&\quad -\frac{1}{\zeta}\left[(\alpha_r+\nu_\theta\alpha_\theta)\frac{d}{d\zeta}\left(\zeta T^*\frac{d\tilde{w}}{d\zeta}\right) + (\alpha_r+\nu_\theta\alpha_\theta)\frac{d^2}{d\zeta^2}(\hat{T}\zeta)\right. \\
&\quad\quad \left.\left. -\left(\frac{\nu_\theta}{\nu_r}\right)(\alpha_\theta+\nu_r\alpha_r)\frac{d\hat{T}}{d\zeta}\right]\right\}.
\end{aligned} \tag{VI.60}$$

$$\frac{d\tilde{e}_1}{d\zeta} + \frac{1-k}{\zeta}\tilde{e}_1 = \left[\frac{(1-\nu_\theta)\alpha_r\nu_r - (1-\nu_r)\alpha_\theta\nu_\theta}{\nu_r\zeta}\right]T^* + (\alpha_r+\nu_\theta\alpha_\theta)\frac{dT^*}{d\zeta} \tag{VI.61}$$

Equations (VI.60), (VI.61) are, therefore, the forms assumed by the Berger's equations for the polar orthotropic elastic plate. In terms of the current set of variables, the boundary conditions along the edge of the plate at $\zeta = 1$ assume the form

(i) Clamped Plate:

$$\tilde{w} = 0,\ \tilde{w}_{,\zeta} = 0, \quad \text{at } \zeta = 1 \tag{VI.62a}$$

(ii) Simply Supported Plate:

$$\begin{cases} \tilde{w} = 0 \\ \tilde{w}_{,\zeta\zeta} + \dfrac{\nu_\theta}{\zeta}\tilde{w}_{,\zeta} + 12(\alpha_r+\nu_\theta\alpha_\theta)\left(\dfrac{a}{h}\right)^2\hat{T} = 0 \text{ at } \zeta = 1 \end{cases} \tag{VI.62b}$$

The general temperature variations $\delta T(r, z)$ in [36] are assumed to be such that the dimensionless forms of (VI.58b), namely, T^* and $\hat{T}$ have the representations

$$\begin{cases} T^* = \sum_{j=0,2,4,\ldots}^{\infty} c_j^* \zeta^j \\ \hat{T} = \sum_{j=0,2,4,\ldots}^{\infty} d_j^* \zeta^j \end{cases} \tag{VI.63}$$

and solutions generated by of the system of Berger's type equations (VI.60), (VI.61) are then sought in the form of a series expansion

$$\tilde{w} = 1 + A_2\zeta^2 + A_4\zeta^4 + \ldots. \tag{VI.64}$$

with the solutions generated by (VI.64) subject to either (VI.62a) or (VI.62b); the results are also specialized to the case of an isotropic circular elastic plate, i.e., $\nu_r = \nu_\theta = \nu$, $\alpha_r = \alpha_\theta = \alpha$, $E_r = E_\theta = E$ and for both types of symmetry computations are effected in [36] for the temperature variation given by (VI.40). While the associated analytical results are too complex to repreduce here, we do depict in Fig. 29, for both the istropic and orthotropic cases, some of the conclusions relating temperature increases to deflections at the center of the plate for various ratios $\dfrac{\alpha_\theta}{\alpha_r}$, $\dfrac{\nu_\theta}{\nu_r}$, and $\bar{T}_0/\bar{T}_1$. It may be observed that, in the case when the temperature gradient through the thickness of the plate is taken into account, the plate starts to deflect at the beginning of heating without exhibiting the Euler buckling phenomenon at a critical temperature. The case of no temperature gradient through the thickness of the plate, which corresponds to $\hat{T} = 0$, is shown in Fig. 29 by the sets of broken curves (depicting the deflection after buckling).

As we have already indicated, critiques and extensions of Berger's approximate method have appeared in several places in the thermal (and load) buckling literature; we content ourselves here with reviewing just two such pieces of work, that of Jones, Mazumdar, and Cheung [28], and that of Banerjee and Datta [29]. The results in [28] cover the case of dynamic (as well as static) behavior of plates at elevated temperatures but we will confine our attention to static situations only; the behavior of viscoelastic plates within the context of Berger's approximate scheme is also discussed in [28] and this analysis will be briefly reviewed in section VII.

In [28], the authors begin by reviewing the system of Berger's approximate equations which are written in the form (compare with (VI.8), (VI.9))

$$K \bigtriangledown^4 w + \lambda^2 \bigtriangledown^2 w = t - \bigtriangledown^2 \frac{M_T}{1-\nu} \tag{VI.65a}$$

$$\frac{N_T}{1-\nu} - \frac{12K}{h^2}\epsilon_1 = \lambda^2 \tag{VI.65b}$$

for isotropic response in rectangular Cartesian coordinates where $M_T = (1-\nu)M^T = \int_{-h/2}^{h/2} \alpha E \delta T z dz$, $N_T = (1-\nu)N^T = \int_{-h/2}^{h/2} \alpha E \delta T dz$, and ϵ_1 is given by (VI.2a); pure thermal buckling corresponds, of course, to the case in which $t \equiv 0$, $M_T = 0$. For pure thermal buckling, therefore, (VI.65a) reduces to

$$K \bigtriangledown^4 w + \lambda^2 \bigtriangledown^2 w = 0 \tag{VI.66}$$

Equation (VI.66) is formally identical to the equation governing the mechanical buckling of a rectangular plate subject to uniform compression, i.e., $N_x = N_y = N$, $N_{xy} = 0$. In order, therefore, for a nontrivial solution to exist we must have $\lambda^2 = N_{cr}$, N_{cr} being the critical buckling load. Taking the corresponding solution for w in the form $w = \Lambda w_b(x,y)$, with Λ an arbitrary parameter, we note that w_b is the deformation due to biaxial loading and, thus, satisfies

$$K \bigtriangledown^4 w_b + \frac{N_{cr}}{K} \bigtriangledown^2 w_b = 0 \tag{VI.67}$$

By nondimensionalizing w_b, so that its maximum value is one, we may interpret Λ as being the maximum plate deflection. By substituting for w in (VI.65b) the temperature deflection curve may now be obtained from

$$N_{cr} = \frac{N_T}{1-\nu} - \frac{12K}{h^2}(u_{,x} + v_{,y} + \frac{1}{2}\Lambda^2 |\bigtriangledown w_b|^2) \tag{VI.68}$$

In fact, for a plate whose edges are restrained in the plane of the plate (VI.68) yields

$$\left(\frac{\Lambda}{h}\right)^2 = \frac{\int_0^{2b}\int_0^{2a}\left\{\frac{N_T}{1-\nu} - N_{cr}\right\} dxdy}{6K\int_0^{2b}\int_0^{2a} |\nabla w_b|^2 dxdy} \tag{VI.69}$$

for a plate occupying the domain $0 \le x \le 2a$, $0 \le y \le 2b$. From (VI.69) it follows that the critical buckling temperature corresponds to

$$\int_0^{2b}\int_0^{2a}\left\{\frac{N_T}{1-\nu} - N_{cr}\right\} dxdy = 0 \tag{VI.70}$$

By denoting the value of N_T which corresponds to the critical temperature T_{cr} as $(N_T)_{cr}$, and employing (VI.70), we may rewrite (VI.29) in the form

$$\left(\frac{\Lambda}{h}\right)^2 = \frac{\int_0^{2b}\int_0^{2a}(N_T - (N_T)_{cr})dxdy}{(1-\nu)6K\int_0^{2b}\int_0^{2a} |\nabla w_b|^2 dxdy} \tag{VI.71}$$

To assess the accuracy of the Berger's approximation technique, the authors [28] consider the rectangular plate described above subject to a temperature variation of parabolic type, i.e.,

$$\delta T(x,y,z) = T_0 + T_1\left[1 - \left(\frac{x-a}{a}\right)^2\right]\left[1 - \left(\frac{y-b}{b}\right)^2\right] \tag{VI.72}$$

The plate is simply supported (note that (VI.72) yields $M^T = 0$) and T_0, T_1 in (VI.72) are constants; for this case

$$w_b(x,y) = \sin\frac{\pi x}{2a}\sin\frac{\pi y}{2b}$$

Comparative values of T_{cr} are taken from the work of Forray and Newman [11]. In Fig. 30 we depict (for aspect ratios 1 and 3, with T_0/T_1 as parameter along the curves) the temperature - deflection relation which is obtained from the definition of N_T and (VI.71) along with the corresponding results in [11]; there appears to be excellent agreement between the results obtained by the two different approaches. Comparisons of the results generated by Berger's method with standard results available for clamped and simply supported circular plates subject to uniform temperature fields, and for long narrow plate strips subjected to an arbitrary temperature variation, are also made in [28]. In fact, for the latter problem, assuming the x-axis to be in the direction of the longest side, and neglecting the component of the deflection in the y-direction in comparison with the components in the x and z-directions, (so that the displacements u, w are regarded as functions of x alone), equations (VI.65b) and (VI.66) become,

$$\begin{cases} K\dfrac{d^4w}{dx^4} + \lambda^2\dfrac{d^2w}{dx^2} = 0 \\ \lambda^2 = \dfrac{N_T}{1-\nu} - \dfrac{12K}{h^2}\left(\dfrac{1}{2}\left(\dfrac{dw}{dx}\right)^2 + \dfrac{du}{dx}\right) \end{cases} \tag{VI.73}$$

which are precisely the same as the equations for thermal deflections of a plate strip as derived by Williams [37].

A highlight of the work presented in [28] is the conclusion that the solution of Berger's equations, for the response of thermally heated plates, represents the first term in a perturbation solution of the generalized von Karman equations; this conclusion is arrived at in the following manner: We write (in rectangular Cartesian coordinates) the governing equations for the classical analysis of thermally heated isotropic plates in the form

$$N_{x,x} + N_{xy,y} = 0 \tag{VI.74a}$$

$$N_{y,y} + N_{xy,y} = 0 \tag{VI.74b}$$

$$K\bigtriangledown^4 w - N_x w_{,xx} - 2N_{xy}w_{,xy} - N_y w_{,yy} = t - \bigtriangledown^2\left(\frac{M_T}{1-\nu}\right) \tag{VI.74c}$$

with (compare with (III.2))

$$\begin{cases} N_x = h\hat{K}(e - \mu\epsilon^0_{yy}) - \dfrac{N_T}{1-\nu} \\ N_y = k\hat{K}(e - \mu e^0_{xx}) - \dfrac{N_T}{1-\nu} \\ N_{xy} = \dfrac{1}{2}\mu h\hat{K}e^0_{xy} \end{cases} \tag{VI.75}$$

where $e = e^0_{xx} + e^0_{yy}$, $\mu = 1 - \nu$, and $e^0_{xx}, e^0_{xy}, e^0_{yy}$ are the usual middle surface strain components, and $\hat{K} = E/(1-\nu^2)$ is the plane stress plate stiffness. Following Schmidt [38] and Schmidt and DaDeppo [39] a solution of the equations (VI.74a,b,c), (VI.75) is sought in the form

$$\begin{cases} u = \sum_{n=0}^{\infty} \mu^n u_n(x,y) \\ v = \sum_{n=0}^{\infty} \mu^n v_n(x,y) \\ w = \sum_{n=0}^{\infty} \mu^n w_n(x,y) \end{cases} \tag{VI.76}$$

By substituting (VI.76) into (VI.74a,b,c) and equating, in the usual manner, the coefficients of equal powers of μ, one obtains the following systems of equations for w_n, u_n, and v_n:

(i) For $n = 0$:

$$\begin{cases} K \nabla^4 w_0 + \left(\dfrac{N_T}{1-\nu} - \hat{K}he_0\right) \nabla^2 w_0 = t - \nabla^2 \left(\dfrac{M_T}{1-\nu}\right) \\ \dfrac{\partial}{\partial x}\left[\hat{K}he_0 - \dfrac{N_T}{1-\nu}\right] = 0 \\ \dfrac{\partial}{\partial y}\left[\hat{K}he_0 - \dfrac{N_T}{1-\nu}\right] = 0 \end{cases} \tag{VI.77}$$

where e_0 is $e = \epsilon^0_{xx} + \epsilon^0_{yy}$, based on the functions $u_0(x,y)$ and $v_0(x,y)$.

(ii) For $n \neq 0$:

$$\begin{cases} K \nabla^4 w_n - \hat{K}he_n \nabla^2 w_n \\ \qquad +\hat{K}h[\epsilon_{(n-1)xx} w_{n,xx} + \epsilon_{(n-1)yy} w_{n,yy} \\ \qquad\qquad +2\epsilon_{(n-1)xy} w_{n,xy}] = 0 \\ \dfrac{\partial e_n}{\partial x} - \dfrac{\partial}{\partial x}\epsilon_{(n-1)xx} + \dfrac{\partial}{\partial y}\epsilon_{(n-1)xy} = 0 \\ \dfrac{\partial e_n}{\partial y} - \dfrac{\partial}{\partial y}\epsilon_{(n-1)yy} + \dfrac{\partial}{\partial x}\epsilon_{(n-1)xy} = 0 \end{cases} \tag{VI.78}$$

where $\epsilon_{(n)xx}$ is ϵ^0_{xx} based on $u = u_n$ and $w = w_n$, etc., while e_n is $\epsilon^0_{xx} + \epsilon^0_{yy}$ based on $u = u_n$, $v = v_n$.

The relations (VI.77) for $n = 0$ may be integrated so as to yield

$$\hat{K}he_0 - \frac{N_T}{1-\nu} = \lambda^2 (\lambda^2 \text{ const.}) \tag{VI.79}$$

If the edges of the plate are restrained in the plane of the plate then the first equation in (VI.77) becomes

$$K \nabla^4 w_0 + \frac{E\alpha T}{1-\nu} h \nabla^2 w_0 = t - \nabla^2 \left(\frac{M_T}{1-\nu} \right) \tag{VI.80}$$

We note that the first relation in (VI.77) and (VI.79) are formally equivalent to (VI.8), (VI.9), i.e., the solution generated by Berger's approximation method is formally the same as that which is generated by the zeroth order perturbation solution of the generalized von Karman system for thermal deflection of an isotropic plate.

Finally, we consider the work of Banerjee and Datta [29]; in this paper a modified energy expression is developed and a new system of governing differential equations, in decoupled form, is obtained for thin plates unergoing large deflections. The equations obtained are tested for circular and square plates with immovable as well as movable edge conditions, under uniform static loads, and an extension to cover the case of thermal loading is outlined. For both movable as well as immovable edges the results obtained appear to be in excellent agreement with other known results for the same test problems considered. The basic motivation for the work presented in [29] appears to be the analysis reported in Nowinski and Ohnabe [27] who establish that the Berger's approximation leads to meaningless results for problems involving movable edge conditions.

For a thin isotropic circular plate of radius b the total potential energy Π under isothermal conditions is given by

$$\begin{aligned} \Pi = \frac{K}{2} \int_0^b \Bigg[\left(\frac{d^2 w}{dr^2} \right)^2 + \frac{2\nu}{r} \frac{dw}{dr} + \frac{1}{r^2} \left(\frac{dw}{dr} \right)^2 \\ + \frac{12}{h^2} \left\{ e_1^2 + 2(\nu - 1) e_2 \right\} \Bigg] r dr - \int_0^b twr dr \end{aligned} \tag{VI.81}$$

where e_1 and e_2 are, respectively, the first and second invariants of the middle surface strains, i.e.,

$$\begin{cases} e_1 = \dfrac{du}{dr} + \dfrac{1}{2}\left(\dfrac{dw}{dr}\right)^2 + \dfrac{u}{r} \\ e_2 = \dfrac{u}{r}\left[\dfrac{du}{dr} + \dfrac{1}{2}\left(\dfrac{dw}{dr}\right)^2\right] \end{cases} \tag{VI.82}$$

and where an axisymmetric deformation has been assumed. It is easily seen that (VI.81) may be rewritten in the form

$$\begin{aligned} \Pi = \frac{K}{2}\int_0^b \Bigg[& \left(\frac{d^2w}{dr^2}\right)^2 + \frac{2\nu}{r}\frac{dw}{dr}\frac{d^2w}{dr^2} + \left(\frac{1}{r}\frac{dw}{dr}\right)^2 \\ & \frac{12}{h^2}\left\{\bar{e}_1^2 + (1-\nu^2)\frac{u^2}{r^2}\right\}\Bigg] r dr - \int_0^b twr dr \end{aligned} \tag{VI.83}$$

with

$$\bar{e}_1 = \frac{du}{dr} + \nu\frac{u}{r} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \tag{VI.84}$$

It is now noted that if the expression $(1-\nu^2)\dfrac{u^2}{r^2}$ in (VI.83) is replaced by $\dfrac{1}{4}\lambda\left(\dfrac{dw}{dr}\right)^4$, with λ a factor depending on ν, decoupling of (VI.83) is possible; carrying out this replacement, and using the usual ideas of the variational calculus, one obtains the following system of decoupled differential equations from the energy functional Π:

$$\nabla^4 w - \frac{12}{h^2}\left(\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right) A r^{\nu-1} - \frac{6\lambda}{h^2}\left(\frac{dw}{dr}\right)^2\left(\nabla^2 w + 2\frac{d^2w}{dr^2}\right) = \frac{t}{K} \tag{VI.85a}$$

with A a constant of integration to be determined from

$$\frac{du}{dr} + \nu\frac{u}{r} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 = A r^{\nu-1} \tag{VI.85b}$$

The decoupled system (VI.85a,b), in rectangular Cartesian coordinates, reads as follows:

$$\begin{aligned} \nabla^4 w - \frac{12}{h^2}A\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) - \frac{6\lambda}{h^2}\Bigg[& \nabla^2 w\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\} \\ +2 & \left\{\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial w}{\partial y}\right)^2\right\}\Bigg] = \frac{t}{K} \end{aligned} \tag{VI.86a}$$

and

$$A = \frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{v}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{VI.86b}$$

To gauge the accuracy of the systems (VI.85a,b) and (VI.86a,b) various sample problems are studied in [29] within the context of an isothermal situation. We will confine our remarks to that case involving the circular plate. For the circular plate the usual boundary conditions are, of course,

$$\begin{cases} w = 0, \ \dfrac{dw}{dr} = 0, \ \text{at } r = b \text{ (clamped)} \\ w = 0, \ \dfrac{d^2w}{dr^2} + \dfrac{\nu}{r}\dfrac{dw}{dr} = 0, \ \text{at } r = b \text{ (simply supported)} \end{cases}$$

For movable edge conditions we have $A = 0$ in (VI.85b) while for an immovable edge $u = 0$ at $r = b$; the deflection function which conforms to these conditions may be taken in the form

$$w = w_0 \left[1 - 2\gamma \frac{r^2}{b^2} + \delta \frac{r^4}{b^4}\right] \tag{VI.87}$$

with $\gamma = \delta = 1$ if the edge at $r = b$ is clamped, while

$$\gamma = \frac{3+\nu}{5+\nu}, \ \delta = \frac{1+\nu}{5+\nu}$$

if the edge at $r = b$ is simply supported. By substituting (VI.87) into (VI.85a,b) an error function $E(r)$ is obtained. The Galerkin procedure then requires that

$$\int_0^b E(r)w(r)r dr = 0 \tag{VI.88}$$

and (VI.88) may be used to obtain the central deflection w_0. Once $w(r)$ has been determined, the constant A may be obtained from (VI.85b) by substituting the expression for $w(r)$ into this relation and then integrating over the area of the plate; the terms which involve the in-plane displacements u and v may be eliminated by employing appropriate expressions for these displacements which are compatible with the boundary conditions. While Berger's method leads to meaningless results when the edge of the plate at $r = b$ is free to move in the plane of the plate, for both the clamped and simple support boundary conditions, numerical data presented in [29] seems to indicate that very accurate results may be obtained through the use of (VI.85a,b) for both a movable as well as an immovable edge.

For a (rectangular plate) subject to a stationary temperature variation of the form

$$\delta T(x, y, z) = T_0(x, y) + g(z)T_1(x, y)$$

(VI.86a) may be easily shown to generalize to

$$\begin{aligned}
\nabla^4 w - \frac{12}{h^2} A \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) - \frac{6\lambda}{h^2} \Bigg[\nabla^2 w \left\{ \left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y^2} \right) \right\} \\
+2 \left\{ \frac{\partial^2 w}{\partial x^2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{\partial^2 w}{\partial y^2} \left(\frac{\partial w}{\partial y} \right)^2 \right\} \Bigg] \\
+12 T_0 \alpha \sqrt{\lambda(1-\nu^2)} \frac{\nabla^2 w}{h^2} = -\frac{E\alpha f(h)}{K(1-\nu)} \nabla^2 T_1
\end{aligned} \tag{VI.89}$$

where $f(h) = \int_{-h/2}^{h/2} z g(z) dz$ and it is assumed that $\int_{-h/2}^{h/2} g(z) dz = 0$. The constant of integration A is determined by the following thermal analogue of (VI.86b):

$$\frac{\partial u}{\partial x} + \nu \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{v}{2} \left(\frac{\partial w}{\partial y} \right)^2 - (1+\nu)\alpha T_0 = A \tag{VI.90}$$

It does not appear that the equations (VI.89), (VI.90) have been applied to any sample problems involving thermal buckling of isotropic rectangular plates; nor does it appear that the analogous equations have been developed for the case of rectilinear orthotropic symmetry.

VII. OTHER ASPECTS OF HYGROTHERMAL/THERMAL BUCKLING

In this section we focus our attention on four specific subareas within the domain of hygroexpansive and thermal buckling, namely, buckling in the presence of imperfections, buckling of plates of variable thickness, and the buckling and postbuckling behavior of both heated viscoelastic and heated plastic plates. All four of these subareas are of especial importance with respect to the cockle problem, i.e., the local hygroexpansive buckling of paper because (i) of the almost certain presence of imperfections in any manufactured paper, (ii) of the variable basis weight associated with all samples of any paper product, (iii) of the fact that within various stress domains paper exhibits both viscoelastic and plastic behavior (e.g., Seth and Page [40]).

A) Hygroexpansive/Thermal Buckling in the Presence of Imperfections

The treatment of hygroexpansive or thermal plate buckling problems in the literature, in the presence of imperfections, is very scant; here we follow the work in Gossard, Seide, and Roberts [2]. We work in rectangular Cartesian coordinates and consider only the case of an isotropic, linearly elastic, rectangular plate which possesses an (initial) out-of-plane deflection $w_i = w_i(x, y)$. In lieu of (I.22), the condition representing strain compatability for a deflected plate with initial imperfections assumes the form

$$\begin{aligned}\frac{\partial^2 \epsilon_{xx}^0}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}^0}{\partial x^2} - 2\frac{\partial^2 \epsilon_{xy}^0}{\partial x \partial y} = (\frac{\partial^2 w}{\partial x \partial y})^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} \\ -(\frac{\partial^2 w_i}{\partial x \partial y})^2 + \frac{\partial^2 w_i}{\partial x^2} \cdot \frac{\partial^2 w_i}{\partial y^2}\end{aligned} \tag{VII.1}$$

where in the case, e.g., of thermal expansion, the strain tensor components $\epsilon_{xx}, \epsilon_{xy}$, and ϵ_{yy} by that are related to the Cartesian stress components σ_{xx}, σ_{xy}, and σ_{yy} by that special case of (I.4) which has the following form for isotropic response:

$$\begin{cases} \epsilon_{xx} &= \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) + \alpha T \\ \epsilon_{yy} &= \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx}) + \alpha T \\ \epsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy} \end{cases} \tag{VII.2}$$

In lieu of (VII.2) we have, in terms of the Airy function Φ, as defined by (I.20),

$$\begin{cases} \epsilon^0_{xx} &= \dfrac{1}{E}(\Phi_{,yy} - \nu\Phi_{,xx}) + \alpha T \\ \epsilon^0_{yy} &= \dfrac{1}{E}(\Phi_{,xx} - \nu\Phi_{,yy}) + \alpha T \\ \epsilon^0_{xy} &= -\dfrac{(1+\nu)}{E}\Phi_{,xy} \end{cases} \tag{VII.3}$$

Substitution of (VII.3) into (VII.1) then yields the following generalization of (III.46) to the case where an imperfection $w_i = w_i(x, y)$ is present in the plate

$$\begin{aligned} \Delta^2\Phi = Eh\Bigg\{ \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \left(\frac{\partial w}{\partial x}\right)^2 \left(\frac{\partial w}{\partial y}\right)^2 \\ -(\frac{\partial^2 w_i}{\partial x \partial y})^2 + \frac{\partial^2 w_i}{\partial x^2}\frac{\partial^2 w_i}{\partial y^2}\Bigg\} - (1-\nu)\Delta N^T \end{aligned} \tag{VII.4}$$

where N^T is given by (1.11a). Equation (VII.4) represents, for an isotropic, linearly elastic, plate in Cartesian coordinates, the first of the generalized von Karman equations modified for the effects of an initial imperfection (deflection). It is easily demonstrated that the second of the pair of generalized von Karman equations in this case has the following form which extends (III.4a):

$$K\Delta^2(w - w_i) = \Phi_{,yy}w_{,xx} - 2\Phi_{,xy}w_{,xy} + \Phi_{,xx}w_{,yy} - \Delta M^T + t \tag{VII.5}$$

with the thermal moment M^T as given by (I.14). In the example to be considered, below, we will assume that $M^T \equiv 0$, $t \equiv 0$ so that there is neither a thermal moment nor a distributed applied force acting normal to the middle surface of the plate. Furthermore, following the analysis in [2], we will assume that the deflections of the plate are merely a magnification of the initial deflections w_i, i.e., if w_c is the net plate center deflection, while w_{i_c} is the plate center deflection which can be attributed to the imperfection, then

$$\frac{w_i}{w} = \frac{w_{i_c}}{w_c} \tag{VII.6}$$

Substitution of (VII.6) into (VII.4) and (VII.5), with $M^T = t = 0$, yields

$$\Delta^2\Phi = Eh\left(1 - \frac{w_{i_c}^2}{w_c^2}\right)\left\{\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right\} - (1-\nu)\Delta N^T \tag{VII.7a}$$

and

$$K\left(1 - \frac{w_{i_c}}{w_c}\right)\Delta^2 w = \Phi_{,yy}w_{,xx} - 2\Phi_{,xy}w_{,xy} + \Phi_{,xx}w_{,yy} \tag{VII.7b}$$

In (VII.7a,b) we now set

$$\begin{cases} E^* = E\left(1 - \dfrac{w_{i_c}^2}{w_c^2}\right) \\ \alpha^* = \dfrac{\alpha}{1 - (w_{i_c}^2/w_c^2)} \\ h^* = h\Big/\sqrt{1 + \dfrac{w_{i_c}}{w_c}} \end{cases} \tag{VII.8}$$

and

$$\Phi^* = \frac{1}{h}\Phi \tag{VII.9}$$

so that $\Phi^*(x,y)$ is the Airy function associated with the stress tensor components σ_{xx}, σ_{xy}, and σ_{yy} instead of the stress resultants N_x, N_y, and N_{xy}. We also assume that $\delta T_{,z} = 0$ so that $N^T = \dfrac{\alpha E h}{1-\nu}\delta T(x,y)$. Employing this latter assumption and (VII.8), (VII.9) in (VII.7a,b), and noting that $\alpha E = \alpha^* E^*$, it is easily seen that the governing equations may be written in the form

$$\Delta^2\Phi^* = E^*\left\{\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right\} - \alpha^* E^* \,\nabla^2\,(\delta T) \tag{VII.10a}$$

$$\left(\frac{K^*}{h^*}\right)\Delta^4 w = \Phi^*_{,yy}w_{,xx} + \Phi^*_{,xx}w_{,yy} - 2\Phi^*_{,xy}w_{,xy} \tag{VII.10b}$$

Equations (VII.10a,b) are identical to the large-deflection equations for thermal buckling of a flat plate with Young's modulus E^*, thermal expansion coefficient α^*, and thickness h^*. If the initial deflection w_i satisfies the same homogeneous boundary conditions as would be satisfied by the deflection of an initially flat plate, then the solutions of (VII.10a,b) are identical with those for a flat plate of the same aspect ratio but with E replaced by E^*, α by α^*, and h by h^*. Thus, in order to analyze an initially imperfect plate, a flat plate having the same aspect ratio may be analyzed with the quantities E, α, and h being left arbitrary; then, everywhere in the resulting expression for the solution, one may replace E by $E(1 - \{w_{i_c}^2/w_c^2\})$, α by $\alpha/(1 - \{w_{i_c}^2/w_c^2\})$, and h by $h\Big/\sqrt{1 + \dfrac{w_{i_c}}{w_c}}$. This relatively simple procedure, introduced in [2], then yields the stresses and deflections for the initially imperfect plate.

For the problem of thermal buckling of a flat, isotropic, simply supported rectangular plate of aspect ratio 1.57, subjected to a tentlike temperature distribution, the relationship between the temperature differential T_0 and the center deflection w_c of the plate was determined, in §V, to be given by (V.27). Noting that the plate thickness h also appears in the expression for the bending stiffness $K = Eh^2/12(1-\nu^2)$, and replacing α and h, respectively, by $\alpha\Big/\left(1 - \dfrac{w_{i_c}^2}{w_2^c}\right)$ and $h\big/\sqrt{1 + (w_{i_c}/w_c)}$ in (V.27) yields the relation

$$\frac{b^2 E\alpha h}{\pi^2 D} T_0 = 5.39\left(1 - \frac{w_{i_c}}{w_c}\right) + 1.12(1-\nu^2)\frac{w_c^2 - w_{i_c}^2}{h^2} \tag{VII.11}$$

The Young's modulus E need not be replaced by E^* in as much as the relation (V.27) is independent of E.

The method described above was employed in [2] to compute curves of center deflection versus average edge compressive stress for the problem of bending of a square, isotropic elastic plate with initial imperfections under applied compressive edge stresses; the curves which resulted from this technique turned out to be in excellent agreement with numerical results obtained from an approximate solution of the von Karman large deflection equations for initially imperfect plates with the agreement existing for all cases in which the initial imperfection was a half-sine wave deflection in both the x and y directions. Experimental and theoretical plate center deflection differences $w_c - w_{i_c}$ have been plotted in Fig. 14 as a function of the temperature differential T_0 with the theoretical curve based on (VII.11); on this graph $\alpha = 0.127 \times 10^{-4} in./in. - {}^\circ F$, $h = 0.25in.$, $b = 11.25in.$, $w_{i_c} = 0.045in.$, and

$\nu = 0.33$ so that (VII.11) has the explicit form

$$T_0 = 194.1\left(1 - \frac{0.045}{w_c}\right) + 573.6(w_c^2 - (0.045)^2) \qquad \text{(VII.12)}$$

with T_0 in degrees Fahrenheit and w_c in inches. We have already referred, in §V, to Fig. 14, in connection with the result (V.27) which corresponds to $w_{i_c} = 0$. To the best of the authors' knowledge the technique described in [2] for approximating the thermal buckling solutions of rectangular, isotropic elastic plates has not been extended to either non-rectangular geometries or to the thermal buckling behavior of polar orthotropic or rectilinearly orthotropic elastic plates.

B) Buckling of Plates of Variable Thickness

Very little work has appeared in the buckling literature which deals specifically with the thermal or hygrothermal buckling of thin plates of variable thickness; the most relevant work in this area would seen to be the papers of Mansfield [41], [42] and the paper of Biswas [43]. In [41] the author generates the extension and flexual equations which govern the elastic behavior of a plate of variable thickness, taking into account temperature variations in the plane of the plate as well as across the thickness of the plate. Also to be found in [41] are general solutions for a rectangular plate whose thickness varies exponentially along the length and for a circular or annular plate whose thickness varies as a power of the plate radius. Mansfield also discusses, in [41], the large-deflection plate equations, including the effects of initial irregularities, i.e., imperfections. A companion paper to [41], which appeared about the same time, is Mansfield's work [42]; this paper discusses the large-deflection analysis of a thin circular lenticular plate whose temperature varies linearly through its thickness. The analysis in [42] covers both the buckled as well as the unbuckled regimes. Finally, in [43], Biswas treats the buckling of a heated annular plate whose thickness varies as the m-th power ($m < 1$) of the radial distance from the center of the plate; the general stability criterion is derived and expressions are obtained for the critical buckling compression as well as the critical buckling temperature. Other work of note in this area includes that of Mendelson and Herschberg [44], who develop approximate methods of plane stress analysis for variable thickness plates with temperature variations in the plane, and Thrun [45] who extended the analysis of Olsson [46], for rectangular plates with a linearly varying rigidity, and Olsson [47], for an arbitrarily loaded circular plate with a quadratically varying rigidity, so as to take into account the effect of temperature variations through the thickness of the plate. We now turn to a description of the analysis in [41]-[43].

In [41], Mansfield introduces the invariant differential operator $\diamond^4$ defined for two sufficiently differentiable functions $f(x,y)$, $g(x,y)$, by

$$\begin{aligned}\diamond^4(f,g) \equiv \frac{1}{2}&\left\{(\nabla^2 f)(\nabla^2 g) + \nabla^2(f\,\nabla^2 g + g\,\nabla^2 f)\right\} \\ -\frac{1}{4}&\{\nabla^4(fg) + f\,\nabla^4 g + g\,\nabla^4 f\}\end{aligned}$$

However, for all sufficiently smooth pairs (f,g) it follows that, in fact,

$$\diamond^4(f,g) = [f,g]$$

where [,] is the usual 'bracket' differential operator

$$[f,g] = f_{,xx}g_{,yy} - 2f_{,xy}g_{,xy} + f_{,yy}g_{,xx} \tag{VII.13}$$

Thus, to the extent possible, in the discussion which follows, below, we will use the bracket [,] notation in lieu of Mansfields 'die' notation $\diamond^4$. In particular, in polar coordinates r,θ,

$$\begin{aligned}\diamond^4(f,g) \equiv [f,g] = f_{,rr}&\left(\frac{1}{r}g_{,r} + \frac{1}{r^2}g_{,\theta\theta}\right) \\ -2&\left(\frac{1}{r}f_{,\theta}\right)_{,r}\left(\frac{1}{r}g_{,\theta}\right)_{,r} \\ +g_{,rr}&\left(\frac{1}{r}f_{,r} + \frac{1}{r^2}f_{,\theta\theta}\right)\end{aligned} \tag{VII.14}$$

and the following special results may be noted:

$$
\begin{aligned}
&(i) \quad [r, g] = \frac{1}{r} g_{,rr} \\
&(ii) \quad [r^2, g] = 2 \nabla^2 g \\
&(iii) \quad [r^n, g] = n(n-1) r^{n-2} \left(\frac{1}{r} g_{,r} + \frac{1}{r^2} g_{,\theta\theta} \right) \\
&\qquad\qquad + n r^{n-2} g_{,rr} \\
&(iv) \quad [r h(\theta), g] = \frac{1}{r} g_{,rr} (h + h_{,\theta\theta}) \\
&(v) \quad [\theta, g] = \frac{2}{r^2} \left(\frac{1}{r} g_{,\theta} \right)_{,r}
\end{aligned}
\tag{VII.15}
$$

In deriving the small deflection equations for a heated, isotropic, rectangular plate of variable thickness Mansfield writes the in-plane equilibrium equations in the form

$$
\begin{cases}
(h\sigma_{xx})_{,x} + (h\tau_{xy})_{,y} = 0 \\
(h\sigma_{yy})_{,y} + (h\tau_{xy})_{,x} = 0
\end{cases}
\tag{VII.16}
$$

where $h = h(x, y)$ and introduces the (stress-component based) Airy function $\phi(x, y)$ which satisfies

$$
\sigma_{xx} = \frac{1}{h} \phi_{,yy}, \sigma_{yy} = \frac{1}{h} \phi_{,xx}, \tau_{xy} = -\frac{1}{h} \phi_{,xy} \tag{VII.17}
$$

For this situation, one could still try to define the usual Airy stress function, i.e., $N_x = \Phi_{,yy}$, etc., where

$$
N_x = \int_{-h/2}^{h/2} \sigma_{xx} dz
$$

but it is no longer true that $N_x = h\sigma_{xx}$; thus, in particular,

$$
\Phi_{,yy} = \int_{-h/2}^{h/2} \frac{1}{h} \phi_{,yy} dz \neq \phi_{,yy} \tag{VII.18}
$$

Of course, because of the structure of the in-plane equations (VII.16), it no longer makes sense to work with the stress resultants N_x, N_y, and N_{xy} in the case of a plate with variable thickness $h(x, y)$. The small deflection stress-strain relations are just those given in (I.4) with $\epsilon_{HT} \equiv \epsilon_T = \alpha\delta T$ where δT is taken, in [41], to be the temperature variation through the thickness of the plate. The strain compatability equation (for small deflections) then assumes, in terms of the Airy function $\phi(x, y)$, the form

$$\begin{aligned} \left\{\frac{1}{h}(\phi_{,xx} - \nu\phi_{,yy})\right\}_{,xx} + \left\{\frac{1}{h}(\phi_{,yy} - \nu\phi_{,xx})\right\}_{,yy} \\ +2(1+\nu)\left(\frac{1}{h}\phi_{,xy}\right)_{,xy} + \alpha E\,\nabla^2(\delta T) = 0 \end{aligned} \tag{VII.19}$$

or, in terms of the bracket differential operator

$$\nabla^2\left(\frac{1}{h}\nabla^2\phi\right) - (1+\nu)\left[\frac{1}{h}, \phi\right] + \alpha E\,\nabla^2(\delta T) = 0 \tag{VII.20}$$

With $t(x, y)$ denoting the applied loading, the third equilibrium equation for small-deflections assumes the usual form, i.e.,

$$M_{x,xx} + M_{y,yy} + 2M_{xy,xy} = -t \tag{VII.21}$$

Using (I.4), with $\epsilon_{HT} \equiv \epsilon_T = \alpha\delta T$, the definitions of the moments M_x, M_y, and M_{xy}, and assuming that $\delta T(z) = T_0 + T_1 z \equiv T_0 + \dfrac{\partial(\delta T)}{\partial z} \cdot z$, i.e., that the temperature varies linearly through the plate thickness, it is easy to derive the moment-curvature relations

$$\begin{cases} w_{,xx} = -\dfrac{12}{Eh^3}(M_x - \nu M_y) - \kappa_T \\ w_{,yy} = -\dfrac{12}{Eh^3}(M_y - \nu M_x) - \kappa_T \\ w_{,xy} = -\dfrac{12(1+\nu)}{Eh^3}M_{xy} \end{cases} \tag{VII.22}$$

where

$$\kappa_T = \alpha\frac{\partial(\delta T)}{\partial z} \equiv T_1 \tag{VII.23}$$

Although not considered in Mansfield [41], it is an easy exercise to show that for a general variation $\delta T(z)$ through the plate thickness

$$\kappa_T = \frac{12\alpha}{h^3}\int_{-h/2}^{h/2} \delta T(z) z dz \tag{VII.24}$$

with $h = h(x,y)$. By solving (VII.22) for M_x, M_y, M_{xy} in terms of $w_{,xx}$, $w_{,yy}$, and $w_{,xy}$, and substituting the results into (VII.21), we obtain

$$\nabla^2 (K \nabla^2 w) - (1-\nu)[K,w] + (1+\nu) \nabla^2 (K\kappa_T) = t \qquad \text{(VII.25)}$$

If, in lieu of (VII.21), we include the middle surface forces in the equilibrium equation, i.e., use

$$\begin{aligned} &M_{x,xx} + M_{y,yy} + 2M_{xy,xy} \\ &\qquad = -(t + \phi_{,xx}w_{,yy} - 2\phi_{,xy}w_{,xy} + \phi_{,yy}w_{,xx}) \end{aligned} \qquad \text{(VII.26)}$$

then, in place of (VII.25) we would clearly obtain

$$\nabla^2 (K \nabla^2 w) - (1-\nu)[K,w] + (1+\nu) \nabla^2 (K\kappa_T) = t + [\phi, w] \qquad \text{(VII.27)}$$

To consider large deflections we replace, of course, the small deflection strain compatability relation by the relation (I.22) where the middle surface strains are given by (I.7); in this case (VII.20) is replaced by

$$\nabla^2 \left(\frac{1}{h} \nabla^2 \phi\right) - (1+\nu) \left[\frac{1}{h}, \phi\right] + \alpha E \nabla^2 (\delta T) + \frac{1}{2}E[w,w] = 0 \qquad \text{(VII.28)}$$

Finally, for a rectangular plate which is initially stress-free, but which possesses an initial irregularity (i.e., imperfection) in the sense that the plate middle surface prior to deflection is given by a relation of the form $z = w_i(x,y)$, the moment-curvature relations are given as in (VII.22) with w replaced by $w - w_i$, e.g.,

$$(w - w_i)_{,xx} = -\frac{12}{Eh^3}(M_x - \nu M_y) - \kappa_T, \quad \text{etc.,}$$

with $w(x,y)$ the final shape of the deflected middle surface of the plate. Also, the compatability equation assumes the form (VII.1). The set of governing equations for large deflections, assuming an initial imperfection $w_i(x,y)$, and also assuming a linear variation of δT through the thickness of the plate, now take on the form

$$\begin{aligned} &\nabla^2 \left(\frac{1}{h} \nabla^2 \phi\right) - (1+\nu) \left[\frac{1}{h}, \phi\right] + \alpha E \nabla^2 (\delta T) \\ &\qquad + \frac{1}{2}E([w,w] - [w_i, w_i]) = 0 \end{aligned} \qquad \text{(VII.29a)}$$

and

$$\begin{aligned} \nabla^2\{K\,\nabla^2\,(w-w_i)\} &- (1-\nu)[K, w-w_i] \\ &+(1+\nu)\,\nabla^2\,(K\kappa_T) \\ &= t + [\phi, w] \end{aligned} \tag{VII.29b}$$

In [41], (VII.29a,b), and their analogoues in terms of polar coordinates, are applied to the solution of small-deflection problems and plane stress problems for rectangular plates with an exponential variation in thickness and to circular or annular plates whose thickness varies as a power of the radius.

Consider a rectangular plate such that

$$h = h_0 \, \exp\,(\gamma\pi y/3a) \tag{VII.30}$$

The plate occupies the domain $0 \le x \le a$, $0 \le y \le b$. Assuming, e.g., that the thickness varies from h_0 to h_b, as y varies from 0 to b,

$$\gamma = \frac{3a}{\pi b} ln \left(\frac{h_b}{h_0}\right) \tag{VII.31}$$

The thickness variation (VII.30) implies a variation in rigidity which is given by

$$K = K_0 \, \exp\,(\gamma\pi y/a) \tag{VII.32}$$

Substitution of (VII.32) into the small-deflection equilibrium equation (VII.25) yields

$$\begin{aligned} \nabla^4 w + \frac{2\gamma\pi}{a}&(w_{,xxy} + w_{,yyy}) \\ &\frac{\gamma^2\pi^2}{a^2}(w_{,yy} + \nu w_{,xx}) = \frac{\bar{t}}{K_0}\,\exp\,(-\gamma\pi y/a) \end{aligned} \tag{VII.33}$$

with

$$\bar{t} = t - (1+\nu)\,\nabla^2\,(K\kappa_T) \tag{VII.34}$$

We suppose that the edges of the plate at $x = 0$ and $x = a$ are simply supported and that the distributed applied loading is such that

$$\bar{t} = e^{\beta\pi y/a} \sum_{m=1}^{\infty} q_m \sin\left(\frac{m\pi x}{a}\right) \tag{VII.35}$$

for $0 \leq x \leq a,\ 0 \leq y \leq b$, with β an arbitrary constant. A solution of (VII.33), (VII.35) is sought, in [41], in the form

$$w = e^{(\beta-\gamma)\pi y/a} w_1(x) + \sum_{m=1}^{\infty} Y_m(y) \sin\left(\frac{m\pi x}{a}\right)$$

with the first term in (VII.35) representing a particular integral of (VII.33); it is easily seen that the particular integral must satisfy the ordinary differential equation

$$\begin{aligned}\frac{d^4 w_1}{dx^4} &+ \frac{\pi^2}{a^2}\left\{2\beta(\beta-\gamma)+\nu\gamma^2\right\}\frac{d^2 w_1}{dx^2} \\ &+\frac{\pi^4}{a^4}\beta^2(\beta-\gamma)^2 w_1 = \frac{1}{K_0}\sum_{m=1}^{\infty} q_m \sin\left(\frac{m\pi x}{a}\right)\end{aligned} \tag{VII.36}$$

whose integration leads to

$$w_1(x) = \frac{a^4}{K_0\pi^4}\sum_{m=1}^{\infty}\frac{q_m}{(m^2+\beta\gamma-\beta^2)^2-\nu m^2\gamma^2}\sin\left(\frac{m\pi x}{a}\right) \tag{VII.37}$$

On the other hand, substitution of the sum in (VII.35) into (VII.33) yields

$$\begin{aligned}a^4\frac{d^4 Y_m}{dy^4} &+ 2\gamma\pi a^3\frac{d^3 Y_m}{dy^3} + \pi^2 a^2(\gamma^2-2m^2)\frac{d^2 V_m}{dy^2} \\ &-2\pi^3\gamma m^2 a\frac{dY_m}{dy} + \pi^4 m^2(m^2-\nu\gamma^2) = 0\end{aligned} \tag{VII.38}$$

Taking the $Y_m(y)$ to have the form

$$Y_m = \sum_{j=1}^{4} A_{m,j}\ \exp\ (r_{m,j}\pi y/a) \tag{VII.39}$$

and substituting into (VII.38) we find that the $r_{m,j}$ must be the roots of the quartic equation

$$r_m^4 + 2\gamma r_m^3 + (\gamma^2-2m^2)r_m^2 - 2\gamma m^2 r_m + m^2(m^2-\nu\gamma^2) = 0 \tag{VII.40}$$

while the $A_{m,j}$ are determined by the boundary conditions along the plates edges at $y = 0$ and $y = b$.

An analysis similar to that outlined above may be effected in order to analyze the plane stress distribution in a rectangular, heated, plate with an exponential variation in thickness.

Indeed, if $h = h_0 \exp(-\lambda \pi y/a)$, a solution of (VII.20), with $\nabla^2(\delta T) = 0$, may be obtained in the form

$$\left\{ \begin{array}{l} \phi = \sum_{m=1}^{\infty} \tilde{Y}_m \left\{ \begin{array}{c} \sin \\ \cos \end{array} \left(\frac{m\pi x}{a} \right) \right\} \\ \tilde{Y}_m = \sum_{j=1}^{4} B_{m,j} e^{s_{m,j}\pi y/a} \end{array} \right. \tag{VII.41}$$

with the $s_{m,j}$ the roots of

$$s_m^4 + 2\lambda s_m^3 + (\lambda^2 - 2m^2)s_m^2)s_m^2 - 2\lambda m^2 s_m + m^2(m^2 + \nu\lambda^2) = 0 \tag{VII.42}$$

For $\nabla^2(\delta T) \neq 0$ a particular integral of (VII.20) may be obtained, e.g., in the case where $\alpha E \nabla^2 T$ has the form

$$\begin{aligned} \alpha E \nabla^2 T &= -\sum_{\beta} e^{\beta\pi y/a} \psi_\beta(x) \\ &= -\sum_{\beta} \left(e^{\beta\pi y/a} \sum_{m=1}^{\infty} \psi_{\beta,m} \sin\left(\frac{m\pi x}{a}\right) \right) \end{aligned} \tag{VII.43}$$

In fact as a particular solution of (VII.20) in this case Mansfield [41] obtains

$$\begin{aligned} \phi_p(x,y) = {} & \frac{h_0 a^4 e^{-\lambda\pi y/a}}{\pi 4} \sum_{\beta} \Bigg(e^{\beta\pi y/a} \\ & \times \sum_{m=1}^{\infty} \frac{\psi_{\beta,m}}{(m^2 + \beta\lambda - \beta^2)^2 + \nu m^2 \lambda^2} \sin\left(\frac{m\pi x}{a}\right) \Bigg) \end{aligned} \tag{VII.44}$$

We now consider the small deflection analysis which is presented in [41] for circular or annular plates whose thickness varies as the power of the radial distance from the center of the plate; specifically, we assume that the plate thickness varies like

$$h(r) = h_o \rho^{\beta/3}, \ \rho = r/r_0 \tag{VII.45}$$

where r_0 is the plate radius and β is an arbitrary constant. The plate rigidity K then varies as $K(r) = K_o \rho^\beta$. Equation (VII.25) may now be written in the following form, which is homogeneous in ρ:

$$\nabla^2 (\rho^\beta \nabla^2 w) - (1-\nu)[\rho^\beta, w] = \bar{t}/K_0 \tag{VII.46}$$

where $\bar{t}$ is given by (VII.34). The solution of (VII.46) corresponding to $\bar{t} = 0$ is given by

$$w = \sum_k R_k \begin{matrix}\sin \\ \cos\end{matrix} \{k\theta\} \tag{VII.47}$$

where

$$R_k = \sum_{i=1}^{4} A_{k,i} \rho^{\gamma_{k,i}} \tag{VII.48}$$

with the $\gamma_{k,i}$ the roots of the bi-quadratic

$$\begin{cases} \Gamma(\gamma_k) = L^2 - L\{2k^2 + \beta(1-\nu)\} \\ \qquad\qquad +k^2\{k^2 + \beta(\beta-1)(1-\nu) - (\beta-2)^2\} \\ L = \gamma_k(\gamma_k + \beta - 2) \end{cases} \tag{VII.49}$$

For a circular plate k assumes successive integral values while for a sector plate subtending an angle θ_0 $k = n\pi/\theta_0$. If $\bar{t}$ in (VII.46) can be expressed in the form of a double summation with respect to arbitrary parameters χ and k, i.e.,

$$\bar{t} = \sum_\chi \sum_k \bar{t}_{\chi,k} \rho^x \begin{matrix}\sin \\ \cos\end{matrix} \{k\theta\} \tag{VII.50}$$

then it is possible to find a particular integral of (VII.46); in fact a particular integral will be given by

$$w_1(r,\theta) = \frac{r_0^4}{K_0} \sum_\chi \sum_k \frac{\bar{t}_{\chi,k} \rho^{x+4-\beta}}{\Gamma(\chi+4-\beta)} \begin{matrix}\sin \\ \cos\end{matrix} \{k\theta\} \tag{VII.51}$$

As an application (actually, an extension) of the approach delineated above Mansfield [41] considers the symmetrical large-deflection bending of a heated circular plate: an unsupported circular plate of lenticular section is subjected to a temperature distribution which varies linearly through the thickness of the plate. For such a plate we may write

$$\begin{cases} h = h_0(1-\rho^2) \\ K = \dfrac{Eh_0^3}{12(1-\nu^2)}(1-\rho^2)^3 \\ \rho = r/r_0 \end{cases} \tag{VII.52}$$

and $\kappa_T = const.$, $w = w(r)$, $\phi = \phi(r)$. The governing differential equations assume the form (VII.27), with $t = 0$ and (VII.28), with $\nabla^2(\delta T) = 0$. For the small deflection version of these equations we would have $w = -\frac{1}{2}\kappa_T r^2$, with r a constant to be determined, and would substitute this form of w, together with (VII.52) into (VII.27), with $t \equiv 0$, so as to obtain

$$(1+\nu)(\kappa_T - \kappa)\,\nabla^2 K = -\kappa\,\nabla^2\phi \tag{VII.53}$$

A solution of (VII.53) is given by

$$\phi = -(1+\nu)\left(\frac{\kappa_T - \kappa}{\kappa}\right) K(r) \tag{VII.54}$$

It may be shown that (VII.54) yields a stress distribution which produces a self-equilibrating system of middle surface forces and which satisfies, therefore, the condition that the edges of the plate are free. Substitution of (III.54), (VII.52), and $w(r) = -\frac{1}{2}\kappa r^2$ into (VII.28), with $\nabla^2(\delta T) = 0$, then shows that $w = -\frac{1}{2}\kappa r^2$, and the expression in (VII.54) for ϕ, yields a solution of the large deflection problem provided

$$\kappa_T - \kappa = \frac{(1-\nu)r_0^4\kappa^3}{2(7+\nu)h_0^3} \tag{VII.55}$$

In [42], Mansfield gives an exact solution for the bending, buckling, and curling of a thin circular plate of lenticular section with a uniform temperature gradient through its thickness; also examined, in [42], is the behavior of such a plate with an initial spherical curvature, when the further possibility of snap-through buckling exists. In the work considered in [42] the plate thickness and rigidity once again vary as in (VII.52). With the temperature gradient variation $\frac{\partial}{\partial z}(\delta T)$ through the thickness constant, each unrestrained element of the plate would assume a uniform spherical curvature $\kappa_T = \alpha\frac{\partial(\delta T)}{\partial z} \equiv \frac{\alpha T_1}{h_0}$ where T_1 is the temperature difference across the thickness of the plate at the center of the plate; also, for a plate subject to such a temperature distribution the deflection of the plate assumes the form

$$w = ar^2\{1 + b\cos 2(\theta - \theta_0)\} \tag{VII.56}$$

with a, b constants depending on the magnitude of κ_T and the initial spherical curvature κ_0. If the plate deforms with rotational symmetry then w assumes the form given in [41], i.e., $w = -\frac{1}{2}\kappa r^2$; otherwise (VII.56) may be written in the form

$$w = -\frac{1}{2}(\kappa_x x^2 + \kappa_y y^2) \tag{VII.57}$$

where κ_x, κ_y are independent of x and y and are the curvatures in the x and y directions, respectively. For a suitable choice of the arbitrary angle θ_0 in (VII.57) one may achieve $\kappa_x \geq \kappa_y$. Thus, the deflection of the plate is completely determined by the curvatures κ_x and κ_y with $\kappa_x = \kappa_y = \kappa$ in the case of rotational symmetry. Another feature of the problem under consideration is that while the magnitude of the middle-surface stresses depends on κ_T and κ_o the distribution of these stresses depends only on r; in fact, the middle surface forces may be derived, in the usual way, from an Airy function ϕ with

$$\begin{cases} h\sigma_{rr} &= N_r = \frac{1}{r}\phi_{,r} \\ h\sigma_{\theta\theta} &= N_\theta = \phi_{,rr} \\ \sigma_{r\theta} &= 0 \end{cases} \tag{VII.58}$$

where

$$\phi \sim (1 - \rho^2)^3 \tag{VII.59}$$

As ϕ varies with r in the same way that K does, and ϕ and K have the same dimensions, we may write that

$$\phi(r) = \beta K(r) \tag{VII.60}$$

so that the distribution of middle-surface stresses is determined once the value of β is known.

The bending stresses σ^b_{xx} and σ^b_{yy} are more conveniently referred to Cartesian coordinates; they vary linearly through the thickness of the plate with their values on the surface $z = \frac{1}{2}h$ given by

$$\sigma^b_{xx} = \frac{6}{h^2}M_x \text{ and } \sigma^b_{yy} = \frac{6}{h^2}M_y \tag{VII.61}$$

The moments M_x, M_y per unit length are given by

$$\begin{cases} M_x = -K\{w_{,xx} + \nu w_{,yy} + (1+\nu)\kappa_T\} \\ M_y = -K\{w_{,yy} + \nu w_{,xx} + (1+\nu)\kappa_T\} \end{cases} \tag{VII.62}$$

so that by combining (VII.61), (VII.62) we obtain

$$\begin{cases} \sigma^b_{xx} = \dfrac{Eh}{2(1-\nu^2)}[\kappa_x + \nu\kappa_y - (1+\nu)\kappa_T] \\ \sigma^b_{yy} = \dfrac{Eh}{2(1-\nu^2)}[\kappa_y + \nu\kappa_x - (1+\nu)\kappa_T] \end{cases} \tag{VII.63}$$

At $\rho = 1$ (the edge of the plate) there are no moments or forces applied; these conditions are automatically satisfied because the variation of K (and ϕ, by virtue of (VII.60)) is such that $K = K_{,r} = 0$ at $\rho = 1$.

It is convenient to introduce the following non-dimensional quantities which are employed by Mansfield in [42]:

$$\begin{aligned} \{\hat{\kappa}_T, \hat{\kappa}, \hat{\kappa}_x, \hat{\kappa}_y, \hat{\kappa}_0\} &= \frac{r_0^2}{h_0}\{\kappa_T, \kappa, \kappa_x, \kappa_y, \kappa_0\} \\ \{\hat{\sigma}_{xx}, \hat{\sigma}_{yy}, \hat{\sigma}_{\theta\theta}\} &= \frac{r_0^2}{Eh_0^2}\{\sigma_{xx}, \sigma_{yy}, \sigma_{rr}, \sigma_{\theta\theta}\} \end{aligned} \tag{VII.64}$$

With a small temperature gradient through the thickness of the plate, the plate deforms into a shallow 'saucer' with constant spherical curvature κ; this spherical curvature is initially governed by small-deflection theory so that $\kappa = \kappa_T$ and the plate is stress free. However, as κ increases, middle-surface stresses are developed which stiffen the plate so that the curvature κ becomes less than κ_T which results in the formation of bending stresses. At a critical value $\kappa_T = \kappa_T^*$ middle-surface stresses dominate the process and for $\kappa_T > \kappa_T^*$ the plate is forced into a shape which is no longer rotationally symmetric. For $\kappa_T >> \kappa_T^*$ the plate approximates a developable surface with parallel generators, i.e., the plate curls up about a diameter. To illustrate the development of this process we use the large-deflection equations for a plate of variable thickness, i.e., (VII.27), with $t \equiv 0$, and (VII.28), with $\nabla^2(\delta T) = 0$, and will, henceforth, refer to these respective specializations of (III.27) and (III.28) as ($\overline{III.27}$) and ($\overline{III.28}$); these equations are coupled and nonlinear.

However, by virtue of the ansatz (VII.60)

$$\begin{aligned} \nabla^2\left(\frac{1}{h}\nabla^2\phi\right) &= \frac{\beta K_0}{h_0 r_0^4}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right) \\ &\quad \times\left\{\frac{1}{1-\rho^2}\left(\frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}\right)(1-\rho^2)^3\right\} \\ &= 144\frac{\beta K_0}{h_0 r_0^4} \end{aligned} \tag{VII.65}$$

In a similar fashion,

$$\begin{aligned} \left[\frac{1}{h},\phi\right] &= \frac{\beta K_0}{h_0 r_0^4}\frac{1}{\rho}\left\{\frac{\partial^2}{\partial\rho^2}\left(\frac{1}{1-\rho^2}\right)\frac{\partial}{\partial\rho}(1-\rho^2)^3\right. \\ &\quad \left.+\frac{\partial^2}{\partial\rho^2}(1-\rho^2)^3\frac{\partial}{\partial\rho}\left(\frac{1}{1-\rho^2}\right)\right\} \\ &= -24\frac{\beta K_0}{h_0 r_0^4} \end{aligned} \tag{VII.66}$$

Also, as $w = -\frac{1}{2}(\kappa_x x^2 + \kappa_y y^2)$, we have

$$\frac{1}{2}[w,w] = \kappa_x\kappa_y (\equiv \text{ const.}) \tag{VII.67}$$

With the assumptions that have been made relative to the form of w and ϕ, ($\overline{III.29}$) assumes the non-dimensional form

$$\lambda\beta + (1+\nu)\hat{\kappa}_x\hat{\kappa}_y = 0 \tag{VII.68}$$

with $\lambda = 2(7+\nu)/(1-\nu)$. Concerning ($\overline{VII.28}$), Mansfield [42] first treats the case in which $\kappa_x = \kappa_y = \kappa$; for that situation

$$\nabla^2 w = -2\kappa \tag{VII.69}$$

while

$$\begin{cases} [K,w] = -\kappa\nabla^2 K \\ [\Phi,w] = -\beta\kappa\nabla^2 K \end{cases} \tag{VII.70}$$

In this case, $(\overline{VII.28})$ is easily seen to reduce to the non-dimensional equation

$$\hat{\kappa}(1+\nu-\beta) = \hat{\kappa}_T(1+\nu) \tag{VII.71}$$

For the situation in which $\kappa_x \neq \kappa_y$, $(\overline{VII.28})$ becomes

$$\{(1+\nu)\kappa_T - (\kappa_x+\kappa_y)\} \nabla^2 K = (\beta+1-\nu)[K,w] \tag{VII.72}$$

which is clearly satisfied if

$$\begin{cases} \hat{\kappa}_x + \hat{\kappa}_y = (1+\nu)\hat{\kappa}_T \\ \beta+1-\nu = 0 \end{cases} \tag{VII.73}$$

By taking $\hat{\kappa}_x = \hat{\kappa}_y = \hat{\kappa}$ in (VII.69) it may be verified that the solution of (VII.68), (VII.71) is given by

$$\beta = -(1+\nu)\hat{\kappa}^2/\lambda \tag{VII.74}$$

with $\hat{\kappa}$ the root of the cubic

$$\hat{\kappa}(1+\hat{\kappa}^2/\lambda) = \hat{\kappa}_T \tag{VII.75}$$

and $\lambda \simeq 20.9$ when $\nu = 0.3$. It is shown, below, that this solution is valid in the range

$$\begin{cases} |\hat{\kappa}_T| \leq \hat{\kappa}_T^* \\ \hat{\kappa}_T^* = \dfrac{2}{1+\nu}\left\{\dfrac{2(7+\nu)}{1+\nu}\right\}^{1/2} \simeq 5.15 \end{cases} \tag{VII.76}$$

For $\hat{\kappa}_T = \hat{\kappa}_T^*$ the plate is about to buckle and (VII.74)-(VII.76) implies that

$$\hat{\kappa} = \left\{\frac{2(7+\nu)}{1+\nu}\right\}^{1/2} \simeq 3.35 \tag{VII.77}$$

Equaiton (VII.77) may be written in dimensional form so as to express the deflection at the edge of the plate in terms of h_0, i.e.

$$w|_{\rho=1} = -\frac{1}{2}\hat{\kappa}h_0 \simeq -1.67h_0 \tag{VII.78}$$

The prebuckling middle surface stresses are given by

$$\begin{cases} \hat{\sigma}_{rr} = \dfrac{(1-\rho^2)\hat{\kappa}^2}{4(7+\nu)} \\ \hat{\sigma}_{\theta\theta} = \dfrac{(1-5\rho^2)\hat{\kappa}^2}{4(7+\nu)} \end{cases} \tag{VII.79}$$

while the bending stresses are

$$\hat{\sigma}^b_{xx} = \hat{\sigma}^b_{yy} = -\frac{1-\rho^2)\hat{\kappa}^3}{4(7+\nu)} \tag{VII.80}$$

Thus, the middle surface stresses vary as the square of the plate curvature while the bending stresses vary as the cube of the plate curvature; the variation of the stress distribution with the magnitude of the temperature gradient is complicated by virtue of the non-linear variation of κ with κ_T. To determine the postbuckling behavior of the plate we solve (VII.68) and (VII.73) so as to obtain

$$\begin{cases} \beta = -(1-\nu) \\ \hat{\kappa}_x = \dfrac{1}{2}(1+\nu)\left\{\hat{\kappa}_T + (\hat{\kappa}_T^2 - \hat{\kappa}_T^{*2})^{1/2}\right\} \\ \hat{\kappa}_y = \dfrac{1}{2}(1+\nu)\left\{\hat{\kappa}_T + (\hat{\kappa}_T^2 - \hat{\kappa}_T^{*2})^{1/2}\right\} \end{cases} \tag{VII.81}$$

The minimum value of $\hat{\kappa}_T^*$ for which the solution (VII.81) exists is $\hat{\kappa}_T^*$ and at this value the solutions represented by (VII.74), (VII.75), and (VII.81) coincide. If $|\hat{\kappa}_T| > \hat{\kappa}_T^*$ then the plate strain energy for the asymmetrical mode of deformation is less than that for the rotationally symmetric mode so that the correct solution is given by (VII.81) and the solution given by (VII.74), (VII.75) then represents an unstable state.

For the postbuckling solution given by (VII.81), the fact that β is constant implies that the middle-surface stresses are independent of κ_T and are given by

$$\begin{cases} \hat{\sigma}_{rr} = \dfrac{1-\rho^2}{2(1+\nu)} \\ \hat{\sigma}_{\theta\theta} = \dfrac{1-5\rho^2}{2(1+\nu)} \end{cases} \tag{VII.82}$$

while the bending stresses assume the form

$$\begin{cases} \hat{\sigma}^b_{xx} = -\dfrac{1}{4}(1-\rho^2)\left\{\hat{\kappa}_T - (\hat{\kappa}_T^2 - \hat{\kappa}_T^{*2})^{1/2}\right\} \\ \hat{\sigma}^b_{yy} = -\dfrac{1}{4}(1-\rho^2)\left\{\hat{\kappa}_T + (\hat{\kappa}_T^2 - \hat{\kappa}_T^{*2})^{1/2}\right\} \end{cases} \tag{VII.83}$$

For $|\hat{\kappa}_T| >> \hat{\kappa}_T^*$, (VII.81) together with (VII.83) yield the asymptotic results

$$\begin{cases} \hat{\kappa}_x \to (1+\nu)\hat{\kappa}_T + \mathcal{O}(1/\hat{\kappa}_T) \\ \hat{\kappa}_y \to 0 + \mathcal{O}(1/\hat{\kappa}_T) \end{cases} \tag{VII.84}$$

and

$$\begin{cases} \hat{\sigma}^b_{xx} \to 0 + \mathcal{O}(1/\hat{\kappa}_T) \\ \hat{\sigma}^b_{yy} \to -\dfrac{1}{2}(1-\rho^2)\hat{\kappa}_T + \mathcal{O}(1/\hat{\kappa}_T) \end{cases} \tag{VII.85}$$

The asymptotic results delineated in (VII.84) and (VII.85) are in agreement with analogous results for inextensional plate theory as given, e.g., in Mansfield [48]. The variation of the principal plate curvatures with the temperature gradient through the thickness of the plate is depicted in Fig. 31 while the variation of both the middle surface stresses and the bending stresses with the temperature gradient through the thickness of the plate is shown in Fig. 32.

For the case in which the circular lenticular plate has a uniform spherical curvature κ_0 when κ_T is zero, and is initially stress-free, the governing differential equations, assuming an initial deflection w_i, are (VII.29a), with $\nabla^2(\delta T) = 0$, and (VII.29b), with $t \equiv 0$, i.e.

$$\begin{aligned} &\nabla^2\left(\frac{1}{h}\nabla^2\phi\right) - (1+\nu)\left[\frac{1}{h},\phi\right] \\ &\quad + \frac{1}{2}E([w,w] - [w_i,w_i]) = 0 \end{aligned} \tag{VII.86a}$$

and

$$\begin{aligned} &\nabla^2(K\,\nabla^2\,(w-w_i)) - (1-\nu)[K, w-w_i] \\ &\quad + (1+\nu)\,\nabla^2\,(K\kappa_T) - [\phi, w] = 0 \end{aligned} \tag{VII.86b}$$

In (VII.86a,b) we may assume that

$$w_i = -\frac{1}{2}\kappa_0 r_0^2$$

Taking, again, trial solutions $\phi = \beta K$, $w = -\frac{1}{2}(\kappa_x x^2 + \kappa_y y^2)$, (VII.86a) reduces to the (non-dimensional) form

$$\lambda\beta + (1+\nu)(\hat{\kappa}_x\hat{\kappa}_y - \hat{\kappa}_0^2) = 0 \tag{VII.87}$$

With $\kappa_x = \kappa_y$, (VII.86b) becomes (for the rotationally symmetric case)

$$\hat{\kappa}(1+\nu-\beta) = (1+\nu)(\hat{\kappa}_0 + \hat{\kappa}_T) \tag{VII.88}$$

while for $\kappa_x \neq \kappa_y$ (VII.86b) assumes the form

$$\begin{aligned}\{(1+\nu)(\kappa_0+\kappa_T) - (\kappa_x+\kappa_y)\}\, \nabla^2 K \\ = (\beta+1-\nu)[K, w]\end{aligned} \tag{VII.89}$$

which is clearly satisfied provided

$$\begin{cases} \hat{\kappa}_x + \hat{\kappa}_y = (1+\nu)(\hat{\kappa}_0 + \hat{\kappa}_T) \\ \beta = \nu - 1 \end{cases}$$

The solution of (VII.87), (VII.88) is given by

$$\beta = -(1+\nu)(\hat{\kappa}^2 - \hat{\kappa}_0^2)/\lambda \tag{VII.90a}$$

with $\hat{\kappa}$ a root of the cubic

$$(\hat{\kappa} - \hat{\kappa}_0)\,\{1 + \hat{\kappa}(\hat{\kappa} + \hat{\kappa}_0)\lambda\} = \hat{\kappa}_T \tag{VII.90b}$$

depending on the relative magnitudes of $\hat{\kappa}_0$ and $\hat{\kappa}_T$, (VII.90b) has either one or three real roots. Following Mansfield [42], we set

$$J = 27\lambda^2(\hat{\kappa}_0 + \hat{\kappa}_T)^2 - 4(\hat{\kappa}_0^2 - \lambda)^3 \tag{VII.91}$$

and note that for $J < 0$, (VII.90b) possesses three real roots; the (algebraically) largest and smallest roots correspond to stable configurations while the middle root is associated with an unstable configuration. When $\hat{\kappa}_T = 0$ there are two stable states for $|\hat{\kappa}_0| > 2\lambda^{1/2}$ which are given by

$$\begin{cases} \hat{\kappa} = \hat{\kappa}_0 \\ \hat{\kappa} = -\dfrac{1}{2}\left\{\hat{\kappa}_0 + (\hat{\kappa}_0^2 - 4\lambda)^{1/2}\right\} \end{cases} \tag{VII.92}$$

It is noted, in [42], that when $\hat{\kappa}$ and $\hat{\kappa}_0 + \hat{\kappa}_T$ are of opposite sign, and $J = 0$, snap-through buckling of the plate will occur and the curvature $\hat{\kappa}$ will jump from the unstable value of $\hat{\kappa} = -\frac{1}{2}\{4\lambda(\hat{\kappa}_0 + \hat{\kappa}_T)\}^{1/2}$ to the stable value of $\hat{\kappa} = \{4\lambda(\hat{\kappa}_0 + \hat{\kappa}_T)\}^{1/2}$. For $J > 0$, (VII.90b) has only one real root but the associated plate configuration is stable only if $K < 0$ where

$$K = \frac{1}{4}(1+\nu)^2(\hat{\kappa}_0 + \hat{\kappa}_T)^2 - \hat{\kappa}_0^2 - \frac{2(7+\nu)}{1+\nu} \tag{VII.93}$$

When $K > 0$, the solution of (VII.87), (??) is given by

$$\begin{cases} \beta = -(1-\nu) \\ \hat{\kappa}_x = \dfrac{1}{2}(1+\nu)(\hat{\kappa}_0 + \hat{\kappa}_T) + K^{1/2} \\ \hat{\kappa}_y = \dfrac{1}{2}(1+\nu)(\hat{\kappa}_0 + \hat{\kappa}_T) - K^{1/2} \end{cases} \tag{VII.94}$$

Therefore, after buckling into a mode which is asymmetrical, the middle-surface forces remain constant, and independent of $\hat{\kappa}_0$; also, for large values of $|\hat{\kappa}_T|$ the plate configuration may be approximated by a developable surface for which

$$\begin{cases} \hat{\kappa}_x \to (1+\nu)(\hat{\kappa}_0 + \hat{\kappa}_T) + \mathcal{O}(1/\hat{\kappa}_T) \\ \hat{\kappa}_y \to 0 + \mathcal{O}(1/\hat{\kappa}_T) \end{cases} \tag{VII.95}$$

The variation of the principal curvatures with the temperature variation through the thickness of the plate is shown in Figs. 33-36 for, respectively, $\hat{\kappa}_0 = \lambda^{1/2}$ and $\hat{\kappa}_0 = 6, 10, 12$; the graphs in Figs. 33-36 possess point symmetry with respect to the point $\hat{\kappa} = 0$, $\hat{\kappa}_T = -\hat{\kappa}_0$.

For $|\hat{\kappa}_0| \leq \lambda^{1/2}$ the possibility of snap-through buckling does not exist; this limiting case is depicted in Fig. 33. When $\hat{\kappa}_T = -\hat{\kappa}_0$ the plate is flat but has zero stiffness. If $|\hat{\kappa}_0| > \lambda^{1/2}$, snap-through buckling will occur at appropriate values of $\hat{\kappa}_T$; however, for $|\hat{\kappa}_0| < 2\lambda^{1/2}$ there exists only one stable state when $\hat{\kappa}_T = 0$. Following snap-through buckling, the plate assumes a symmetrical mode in Fig. 35 but an asymmetrical mode in Fig. 6.

The last work we have made reference to in this section, on the buckling of heated plates with variable thickness, is that of Biswas [43]; this work, which is concerned with the buckling of a heated, thin annular (circular) plate appears to posses a serious flaw which will be noted, below. Biswas [43] considers an annular (circular) plate whose thickness varies as the m^{th} power of the radial distance from the center of the plate. The edges of the plate are restrained from in-plane movement and the plate is subjected to uniform compression along its edges as well as to a stationary temperature distribution of the form

$$T(r,z) = \tau_0(r) + z\tau(r) \tag{VII.96}$$

so that (pure) buckling will occur for $\tau \equiv 0$, $\tau_0 \neq 0$. As a consequence of (VII.97), $M_T = 0$; therefore, with $\phi(r) = -w'(r)$

$$\begin{cases} M_r = K\left(\phi'(r) + \dfrac{\nu}{r}\phi(r)\right) \\ M_\theta = K\left(\nu\phi'(r) + \dfrac{1}{r}\phi(r)\right) \\ K = K(r) \equiv \dfrac{Eh^3(r)}{12(1-\nu^2)} \end{cases} \tag{VII.97}$$

while

$$N_r = \frac{Eh}{1-\nu^2}\left(\frac{du}{dr} - \frac{\nu}{r}u\right) - \frac{N_T}{1-\nu} \equiv \frac{-N_T}{1-\nu} \tag{VII.98}$$

as the radial displacement $u = u(r) \equiv 0$, in the plane of the plate, due to the restraint imposed along the edge of the plate. In (VII.98) we have, of course,

$$\begin{aligned} N_T &= \alpha E \int_{-h/2}^{h/2} \tau_0(r)dz \\ &\equiv \alpha E \tau_0(r) h(r) \end{aligned} \tag{VII.99}$$

The thickness variation for the plate is taken to be of the form

$$h(r) = h_0 r^m, \ m < 1 \tag{VII.100}$$

in which case

$$K(r) = \frac{Eh_0^3}{12(1-\nu^2)} r^{3m} \equiv K_0 r^{3m}, \ m < 1 \tag{VII.101}$$

Setting $\beta = \dfrac{b}{a}$, and $\rho = \dfrac{r}{a}$, (so that the outer and inner boundaries of the annular plate correspond, respectively, to $\rho = 1$ and $\rho = \beta$), and employing (VII.94) in the equilibrium equation

$$M_r + rM_r' - M_\theta + hN_r\phi = 0 \tag{VII.102}$$

Biswas [43] arrives at the following ordinary differential equation for $\phi = \phi(r)$:

$$\rho^2 \frac{d^2\phi}{d\rho^2} + (3m+1)\rho\frac{d\phi}{d\rho} + \left(\frac{h_0 N_r}{K_0} a^{-2m+2} \rho^{-2m+2} + 3m\nu - 1\right)\phi = 0 \tag{VII.103}$$

whose solution (it is claimed in [43]) has the form

$$\phi(\rho) = \rho^{-3m/2}\left\{AJ_\mu(\lambda\rho^{-m+1}) + BY_\mu(\lambda\rho^{-m+1})\right\} \tag{VII.104}$$

with

$$\left\{\begin{array}{l} \lambda = \dfrac{1}{(1-m)}\left(\dfrac{h_0 N_r}{K_0} a^{-2m+2}\right)^{1/2} \\ (1-m)^2\mu^2 = \dfrac{9}{4}m^2 - 3\nu + 1 \end{array}\right. \tag{VII.105}$$

The specification of the roots λ is then sought for the case in which the outer plate and inner plate boundary conditions are given by

$$\phi = 0, \ \text{for } \rho = 1, \beta; \ w = 0, \ \text{for } \rho = 1 \tag{VII.106}$$

Using the boundary date (VII.106) in (VII.104), Biswas [43] obtains as the condition from which the critical compression (and, thus, the critical temperature) can be determined for different values of m, ν, and β

$$J_\mu(\lambda)Y_\mu(\lambda\beta^{1-m}) - J_\mu(\lambda\beta^{1-m})Y_\mu(\lambda) = 0 \tag{VII.107}$$

Unfortunately, λ as given by (VII.105) is a function of r because $N_r \equiv -\frac{1}{1-\nu}N_T = \frac{-\alpha E}{1-\nu}\tau_0(r)h(r)$.

Even if $\tau_0 \equiv$ const., we would still have a situation in which $N_r = N_r(r)$ because of the thickness variation of the plate with radial distance. As $\lambda = \lambda(r)$, (VII.103) while an equation of Bessel type, is a variable coefficient (not a constant coefficient) ordinary differential equation for which the standard solution (VII.104) does not hold.

C) Viscoelastic and Plastic Buckling

While various papers have appeared in the literature which have investigated the nature of the stress distributions in both viscoelastic and plastic plates at elevated temperatures, few studies exist that are concerned with the deflection or buckling behavior of viscoelastic or plastic plates which are thermally loaded. For viscoelastic plates, or plates subject to temperature-dependent creep buckling, notable contributions have been made by Ross and Berke [49], Jones and Mazumdar [28], and Das [50] which will be summarized, below. The most significant piece of work in the literature, which is related to the thermal buckling of plates in the plastic regime, appears to be the paper of Williams [51]; in this paper the field equations associated with Neale's variational theorem [52], [53] are developed and applied to the problem of thermal buckling and postbuckling of constrained rectangular plates; the resulting equations are a generalization of von Karman's equations in rate form and among the results, which will be described in this section, is the fact that, in the immediate vicinity of a critical point, the theory predicts a substantial reduction of the buckling temperature due to plasticity effects. We begin our analysis of these problems by first looking at viscoelastic and temperature-dependent creep bucking.

In [49] the authors employ the so-called Norton-Bailey power law for material creep to predict the time-dependent lateral deflection of flat rectangular plates with a through-thickness steady-state temperature distribution. The usual Norton-Bailey or power creep law (e.g., Norton [54]) has the form

$$\dot{\epsilon} = k\sigma^n \tag{VII.108}$$

where $\dot{\epsilon}$ is the creep strain rate, k is the creep constant, and σ is the stress. In considering temperature variations, one modifies (VII.108) so that it assumes the form given by Maxwell's law [55], i.e.,

$$\dot{\epsilon} = ke^{-H/RT}\sigma^n \tag{VII.109}$$

with H the creep activation energy, R the universal gas constant, and T the absolute temperature. To study the creep buckling of plates, (VII.109) is replaced by the two-dimensional equations

$$\dot{\epsilon}_i = \frac{1}{2}k_n e^{-H/RT} J_2^{n-1}(2\sigma_i - \sigma_j) \tag{VII.110}$$

with the subscripts i, j denoting xx, yy in cyclic substitution, σ_{xx}, σ_{yy} being the usual in-plane stresses in the x and y directions, respectively, while J_2 is the stress invariant

$$J_2 \equiv \sigma_{xx}^2 + \sigma_{yy}^2 - \sigma_{xx}\sigma_{yy} \tag{VII.111}$$

In [49] a comparison of creep buckling predictions is made for the two cases where the creep exponent n in (VII.110) assumes the values $n = 3$ and $n = 5$. For a simply supported flat rectangular plate, such as is depicted in Fig. 37, it is assumed that the deflected shape may be adequately represented by a half-wave cosine function in each of two perpendicular directions at all times, in which case the plate behavior is determined by the plate deflection at the center of the plate. If the plate possesses an initial imperfection given by its value at the plate center, say, w_i^0 and a temperature differential is applied between the sides of the plate, in the thickness direction, the center deflection w^0 due to that temperature differential may, as in [56], be approximated by

$$w_0^T = \frac{\alpha(T_0 - T_i)(1+\nu)ab}{\pi^3 h} \tag{VII.112}$$

where T_i, T_0 are, respectively, the inner and outer plate temperatures. The central deflection after the application of a through-thickness temperature differential, but prior to the application of in-plane stresses, is taken to be given by

$$w_0^c = w_i^0 + w_0^T \tag{VII.113}$$

In [49] an axial force at time $t = 0$ is applied to the plate. The immediate deflection w_0^p, which is attained upon adding the axial load, has been determined, in [57], to have the value

$$w_0^p = w_0^c \left(\frac{\sigma_E}{\sigma_E - \sigma}\right) \tag{VII.114}$$

with σ the applied axial stress and σ_E the Euler buckling stress; for a plate (which is isotropic, as we have assumed here) subject to an in-plane compressive load

$$\sigma_E = \frac{\pi^2 E}{12(1-\nu^2)} \left[\frac{b}{a} + \frac{a}{b}\right]^2 \frac{h^2}{a^2} \tag{VII.115}$$

If the applied stress σ is larger than the material yield stress that stress is used in (VII.114) in lieu of σ_E. The stresses and lateral displacements in [49] are assumed to be represented as products of $\cos\dfrac{\pi x}{a} \cos\dfrac{\pi y}{b}$, e.g.,

$$w(t) = w_0(t) \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} \tag{VII.116}$$

where $w_0(t)$, the deflection of the center of the plate at time t, is taken to be the sum of a small deflection $w_s(t)$ and a large deflection $w_l(t)$; it has been shown, in [58], that for the creep exponent $n = 3$ these components of the plate deflection are given by the following expressions in which $\Gamma = b^2/a^2$ and k_3 is the material creep constant for $n = 3$:

$$ln\left[\frac{w_s(t)}{w_0^p}\right] = \frac{36 k_3 \sigma^3 t e^{-2H/3RT_0}[e^{-H/3RT_0} + e^{-H/3RT_i}]}{h^2(2\Gamma^2 + 2\Gamma + 1]\left(1 + \exp\left\{\frac{-H}{3R}\left(\frac{1}{T_0} - \frac{1}{T_i}\right)\right\}\right)} \left(\frac{b}{\pi}\right)^2 \tag{VII.117}$$

and

$$w_l(t) = \frac{4\pi}{3} \frac{h^2 w_0^p(\Gamma^2 + \Gamma + 1)}{\sqrt{\frac{16h^2}{9}\pi^2(\Gamma^2 + \Gamma + 1) - 81 b^2 k_3 w_0^{p2} \sigma^3 t \left(e^{-H/RT_0} + e^{-H/RT_i}\right)}} \tag{VII.118}$$

For creep exponent $n = 5$, on the other hand,

$$ln\left[\frac{w_s(t)}{w_0^p}\right] = \frac{2160k_5\sigma^5 t}{h^2} \cdot \frac{1}{(10\Gamma^2 + 10\Gamma + 4)}$$

$$\times \left\{ \frac{e^{-4H/5RT_0}\left(e^{-H/5RT_0} + e^{-H/5RT_i}\right)}{\left[1 + e^{-H/5R\left(\frac{1}{T_0} - \frac{1}{T_i}\right)}\right]^4} \right\} \left(\frac{b}{\pi}\right)^2 \qquad \text{(VII.119)}$$

and

$$\frac{1}{w_0^{p4}} - \frac{1}{w(t)^4} = \frac{18,225}{32} \cdot \frac{k_5\sigma^5 t}{h^6} \cdot \frac{1}{(\Gamma^2 + \Gamma + 1)^3}$$

$$\times \left\{e^{-H/RT_0} + e^{-H/RT_i}\right\} \left(\frac{b}{\pi}\right)^2 \qquad \text{(VII.120)}$$

The predictions embodied in (VII.117)-(VII.120) were studied, numerically, in [49] for specific values of the various parameters involved, e.g., $k_3 = 3.05 \times 10^8/MPa \cdot$ sec and $k_5 = 5.71 \times 10^5/MPa \cdot$ sec. In Figs. 38-40 we indicate the type of predictions which follow from the work in [49], for the lateral deflections as functions of time, for applied axial stresses of 6.9, 13.8, and 34.5 MPa, respectively; these figures, for given values of ΔT and σ, compare the predictions with respect to lateral deflection which are made by the creep power law (VII.110) when the creep exponent n is taken as either 3 or 5. All the curves in Figs. 38-40 display the familiar phenomena of increasing strain rate with time. In Fig. 41 we indicate comparative results for the predictions of the time until the creep deflection reaches a specified value for the creep exponents $n = 3$ and $n = 5$; the chosen value of the fixed creep deflection in Fig. 41 is $5.08 \times 10^{-3}m$ (which is approximately equal to the plate thickness). The range of applicability of the results in Figs. 38-41 is subject to question as they correspond to material properties that were adopted for a constant mean temperature of the plate; in reality, there is ample experimental evidence to suggest that these properties may be strongly dependent on temperature.

In [28], Jones, Mazumdar, and Cheung have considered the small amplitude response of a thermo-rheologically simple viscoelastic plate; in this case the model is such that (VII.27) is replaced by

$$K(p)\,\nabla^4 w + \frac{N_T}{1-\nu(p)}\,\nabla^2 w = t - p^2(\rho h w) - \frac{\nabla^2 M_T}{1-\nu(p)} \tag{VII.121}$$

with $p \equiv \dfrac{\partial}{\partial t}$ and $K(p), \nu(p)$ the viscoelastic time operators corresponding to the flexual rigidity K and Poisson's ratio ν for the elastic case. Equation (VII.121) is identical to the governing equation for the analysis of a viscoelastic plate with in-plane loads

$$N_x = N_y = N_T/(1-\nu(p))$$

and transverse load

$$t - \nabla^2 M_T/(1-\nu(p)),$$

the derivation following the analysis in Deleeuw and Mase [59]. If $t = \nabla^2 M_T = 0$, and inertia effects are negligible, then a deflection $w(x,y,t)$ may be sought which is separable in time and space, i.e.,

$$w(x,y,t) = W(x,y)T(t) \tag{VII.122}$$

where $W(x,y)$ satisfies

$$\nabla^4 W - C\,\nabla^2 W = 0 \tag{VII.123}$$

and where $T(x)$, which represents the magnitude of the deflection, satisfies

$$K(\rho)T(t) - T(t)\frac{N_T}{C(1-\nu(p))} = 0 \tag{VII.124}$$

with c representing a separation constant. Thus, the mode shapes are time independent and only the amplitudes vary with time. If $N_T = \text{const.}$ then it may be shown that there exist upper and lower critical temperatures where the lower critical temperature corresponds to a zero deflection rate while the upper critical temperature corresponds to an infinite deflection. The lower critical temperature is determined by setting $p = 0$ in (VII.124) which yields

$$C = \frac{N_T}{(1-\nu(0))K(0)} \tag{VII.125}$$

Using (VII.125), (VII.123) becomes

$$\nabla^4 W - \frac{N_T}{K(0)(1-\nu(0))} \nabla^2 W = 0 \tag{VII.126}$$

In a similar manner, the upper critical temperature is determined by setting $p = \infty$; this subsequently yields, in lieu of (VII.126),

$$\nabla^4 W - \frac{N_T}{K(\infty)(1-\nu(\infty))} \nabla^2 W = 0 \tag{VII.127}$$

with the notation employed being the same as that in [59]. When the boundary conditions for the plate contain the Poisson's ratio ν (as in the case for a plate with either free or simply supported edges) then in solving (VII.126) or (VII.127) one simply replaces ν by $\nu(0)$ or $\nu(\infty)$, respectively.

For both Kelvin and Maxwell type viscoelastic materials, the upper and lower critical buckling loads are expressible in terms of the elastic critical loads with the correlations being indicated in Table 3. It may be shown that the physical interpretation of the critical temperatures is the same as for the critical buckling loads discussed in [59]. If

$$N_T < \text{ lower critical value}$$

then the deflection decreases with time. If

$$N_T = \text{ lower critical value}$$

then the deflection is constant. For N_T such that

$$\text{lower critical value } < N_T < \text{ upper critical value}$$

the deflection increases with time and, finally, if

$$N_T = \text{ upper critical value}$$

then the deflection is immediately infinite.

In Das [50] the equation governing the deflections of thermally loaded viscoelastic plates is derived as follows: using standard tensor notation, the constitutive relations and kinematic relations are written, following Nowacki [8], as

$$\begin{cases} \sigma_{ij} = \dfrac{E}{1-\nu^2}\left\{(1-\nu)\epsilon_{ij} + [\nu\epsilon_{kk} - (1+\nu)\alpha_t x_3 \tau]\delta_{ij}\right\} \\ \epsilon_{ij} = -x_3 w_{,ij} \end{cases}$$

where $w_{,ij} = \dfrac{\partial^2 w}{\partial x_i \partial x_j}$ and we sum, as usual, on repeated indices. Thus, with the understanding that ν is time-dependent,

$$\sigma_{ij} = \frac{-Ex_3}{1-\nu^2}\left\{(1-\nu)w_{,ij} + [\nu w_{,kk} + (1+\nu)\alpha_t \tau]\delta_{ij}\right\} \tag{VII.128}$$

As the bending moments are given by

$$M_{ij} = \int_{-h/2}^{h/2} x_3 \sigma_{ij} dx_3$$

we have, using (VII.128),

$$M_{ij} = -K\left\{(1-\nu)w_{,ij} + [\nu w_{,kk} + (1+\nu)\alpha_t \tau]\delta_{ij}\right\} \tag{VII.129}$$

with K the usual bending stiffness. Assuming that $M_{ij,ij} = 0$, i.e., ignoring the influence of the middle surface forces, we obtain as the differential equation for the deflection of a thermally loaded viscoelastic plate

$$\nabla^4 w + (1+\nu)\alpha_t \nabla^2 \tau = 0 \tag{VII.130}$$

or

$$\nabla^4 w + m(t) \nabla^2 \tau = 0 \tag{VII.131}$$

with

$$m(t) = (1+\nu)\alpha_t \tag{VII.132}$$

We note that the temperature distribution in the plate, $T(x_1, x_2, x_3)$, has been taken in [50] to have the form

$$T = \tau_0(x_1, x_2) + x_3\tau(x_1, x_2) \tag{VII.133}$$

and that the situation depicted in Fig. 42 applies, i.e.,

$$\begin{cases} T_1(x_1, x_2) = T(x_1, x_2, \frac{h}{2}) \\ T_2(x_1, x_2) = T(x_1, x_2, -\frac{h}{2}) \end{cases} \tag{VII.134}$$

If θ_1 and θ_2 are, respectively, the temperatures of the medium immediately below and above the plate, then the Newton-type boundary conditions

$$\begin{cases} \lambda \dfrac{\partial T}{\partial x_3}\Big|_{x_3=\frac{h}{2}} = \lambda_1(\theta_1 - T_1) \\ \lambda \dfrac{\partial T}{\partial x_3}\Big|_{x_3=-\frac{h}{2}} = -\lambda_1(\theta_1 - T_1) \end{cases} \tag{VII.135}$$

apply where λ is the plate thermal conductivity. From the equation of heat conduction, and the assumptions listed, above, we obtain for the stationary temperature field τ the following equation (assuming the absence of a heat source):

$$\nabla^2 \tau - \frac{12}{h^2}(1+\epsilon)\tau = -\frac{12}{h^3}(\theta_1 - \theta_2) \tag{VII.136}$$

where $\epsilon = h\lambda_1/2\lambda$, or

$$\nabla^2 \tau - k^2\tau = -\beta \tag{VII.137}$$

with

$$k^2 = \frac{12}{h^2}(1+\epsilon) \text{ and } \beta = \frac{12}{h^3}(\theta_1 - \theta_2) = \text{ const.} \tag{VII.138}$$

We note that, following the analysis in [50], the fourth order equation (VII.131) may be replaced by two equations of the second order: using the fact that

$$-K(1+\nu)\nabla^2 w = M_{11} + M_{22}$$

or

$$\nabla^2 w = -\frac{M}{K}; \; M = \frac{M_{11} + M_{22}}{(1+\nu)} \tag{VII.139}$$

From (VII.131) we then obtain

$$\nabla^2 M = m(t) K \nabla^2 \tau \tag{VII.140}$$

so that (VII.131) may be replaced by (VII.139), (VII.140).

Equations (VII.139) and (VII.140) may be simplified for the special case of a simply supported polygonal plate, i.e., the solution of (VII.140) is given by

$$M = m(t) K \tau \tag{VII.141}$$

as M and τ are both zero along the edges of a simply supported plate. Thus, by virtue of (VII.139) and (VII.141), it follows that it is sufficient to solve

$$\nabla^2 w + m(t)\tau = 0 \tag{VII.142}$$

subject to the condition that $w = 0$ along the edges of the plate. Laplace transforming (VII.142) we obtain

$$\nabla^2 \tilde{W} + \frac{\tilde{m}(s)}{s}\tau = 0 \tag{VII.143}$$

with

$$\begin{cases} \tilde{W} = \int_0^\infty w e^{-st} dt \\ \tilde{m}(s) = (1+\tilde{\nu})\alpha_t \end{cases} \tag{VII.144}$$

s being the transform parameter. In equations (VII.137) and (VII.143) we introduce the complex co-ordinate system given by

$$x_3 = x_1 + i x_2, \; \bar{x}_3 = x_1 - i x_2 \tag{VII.145}$$

thus reducing these equations to the respective forms

$$4\frac{\partial^2 \tau}{\partial x_3 \partial \bar{x}_3} - k^2 \tau = -\beta \tag{VII.146a}$$

$$4\frac{\partial^2 \tilde{W}}{\partial x_3 \partial \bar{x}_3} + \frac{\tilde{m}(s)}{s}\tau = 0 \tag{VII.146b}$$

We now let $x_3 = f(\xi)$ be the function which maps the domain of the plate onto the unit circle so that, in the system of coordinates $(\xi, \bar{\xi})$, (VII.146a,b) reduce to

$$4\frac{\partial^2 \tau}{\partial \xi \partial \bar{\xi}} - k^2 \tau \frac{dx_3}{d\xi}\frac{d\bar{x}_3}{d\bar{\xi}} = -\beta \frac{dx_3}{d\xi}\frac{d\bar{x}_3}{d\bar{\xi}} \tag{VII.147a}$$

$$4\frac{\partial^2 \tilde{W}}{\partial \xi \partial \bar{\xi}} + \frac{\tilde{m}(s)}{s}\tau \frac{dx_3}{d\xi}\frac{d\bar{x}_3}{d\bar{\xi}} = 0 \tag{VII.147b}$$

with $\xi = re^{i\theta}$, $\bar{\xi} = re^{-i\theta}$. Assuming that

$$\frac{dx_3}{d\xi} = \frac{d\bar{x}_3}{d\bar{\xi}} = a_1 = \text{ const.} \tag{VII.148}$$

we obtain from (VII.147a)

$$\frac{\partial^2 \tau}{\partial \xi \partial \bar{\xi}} - \mu^2 \tau = -\frac{\beta a_1^2}{4}; \; \mu^2 = \frac{k^2 a_1^2}{4} \tag{VII.149}$$

The closed form solution of (VII.149) is given in the form

$$\begin{aligned} \tau &= AI_0(2\mu\sqrt{\xi\bar{\xi}}) + \frac{\beta a_1^2}{4\mu^2} \\ &= AI_0(2\mu r) + \frac{\beta a_1^2}{4\mu^2} \end{aligned} \tag{VII.150}$$

with $I_0(2\mu r)$ the modified Bessel function of zeroth order and A a constant to be determined by the boundary condition $\tau = 0$ at $r = 1$. A direct computation shows that

$$A = -\frac{\beta a_1^2}{4\mu^2} \cdot \frac{1}{I_0(2\mu)} \tag{VII.151}$$

in which case

$$\tau = \frac{\beta a_1^2}{4\mu^2}\left[1 - \frac{I_0(2\mu r)}{I_0(2\mu)}\right] \tag{VII.152}$$

To solve (VII.147b) we again apply the assumption (VII.148) in which case (VII.147b) reduces to

$$\frac{\partial^2 \tilde{W}}{\partial \xi \partial \bar{\xi}} + \gamma^2 \tau = 0 \tag{VII.153}$$

with

$$\gamma^2 = \frac{\tilde{m}(s)}{4s} a_1^2 \tag{VII.154}$$

Substituting for τ in (VII.153) from (VII.152) we obtain

$$\begin{aligned} \frac{\partial^2 \tilde{W}}{\partial \xi \partial \bar{\xi}} &= -\gamma^2 \frac{\beta a_1^2}{4\mu^2} \left[1 - \frac{I_0(2\mu r)}{I_0(2\mu)} \right] \\ &= P + Q I_0(2\mu r) \end{aligned} \tag{VII.155}$$

where

$$P = -\frac{\gamma^2 \beta a_1^2}{4\mu^2}, \; Q = -\frac{P}{I_0(2\mu)} \tag{VII.156}$$

The solution of (VII.155) is of the form

$$\tilde{W} = B + Pr^2 + \frac{Q}{\mu^2} I_0(2\mu r) \tag{VII.157}$$

with B determined by the boundary condition $\tilde{W} = 0$ at $r = 1$; applying this boundary condition we have

$$B = -P - \frac{Q}{\mu^2} I_0(2\mu) \tag{VII.158}$$

in which case

$$\tilde{W} = -P - \frac{Q}{\mu^2} I_0(2\mu) + Pr^2 + \frac{Q}{\mu^2} I_0(2\mu r) \tag{VII.159}$$

The maximum of $\tilde{W}$ occurs at $r = 0$ and has the value

$$(\tilde{W})_{\max} = \frac{\tilde{m}(s)}{s} \cdot \frac{\beta a_1^4}{16\mu^2} \left[1 - \frac{1}{\mu^2} + \frac{1}{\mu^2 I_0(2\mu)} \right] \tag{VII.160}$$

By taking the inverse Laplace transform of (VII.160) the maximal plate deflection is obtained.

Finally we turn our attention to the problem of thermal buckling of elastic-plastic plates; the most fundamental and comprehensive treatment of this problem, to date, appears to be the work of Williams [51] which is based on a variational formulation of the rate problem in elastoplasticity by Neale [52], [53] in which both the strain and stress rates can be varied independently. In the treatment developed in [51], which we will outline, below, the field equations that follow from Neale's original analysis [52] are employed to study the thermal buckling of constrained rectangular plates. It is worth noting that, in contrast with the constitutive assumptions of Neale [51], who uses a J_2-incremental theory, and employs the Ramberg-Osgood relation to describe the hardening behavior, several authors have approached the same basic problem using deformation theories of plasticity; these include the work of Mayers and Nelson [60], who use a modified form of Reissnerś variational principle and Turvey [61] who employs a constitutive equation that was proposed by Myszkowski [62].

One advantage of the approach in Neale's work [52] is that when the Ramberg-Osgood hardening exponent is set equal to 3.0 the hardening parameter becomes a constant and this is the case considered by Williams in [51]; because the interest in [51] is in thermal behavior, a slight modification of the energy potential, as suggested by Neale in [52], is introduced by Williams [51]. The buckling and post buckling equations in [51] are derived following the approach in Danielson [63].

Following the Lagrangian formulation proposed by Neale [52], the displacement components and relevant Kirchhoff stress components in [51] assume the form (see Fig. 43)

$$\begin{cases} u_1 = u(x,y) - zw_{,x}(x,y) \\ u_2 = v(x,y) - zw_{,y}(x,y) \\ u_3 = w(x,y) \end{cases} \tag{VII.161}$$

and

$$P_{ij} = \frac{N_{ij}(x,y) + \dfrac{12z}{h^2} M_{ij}(x,y)}{h} \tag{VII.162}$$

The equations of equilibrium and the constitutive relations governing u, v, w, N_{ij}, etc., are obtained from the requirement that $\delta\mathcal{I}^{(0)} = 0$ where the Lagrangian form of the functional $\mathcal{I}^{(0)}$ is

$$\begin{aligned} \mathcal{I}^{(0)} = \int_{V_0} \left\{ \dot{P}_{ij}\dot{E}_{ij} + \frac{1}{2} P_{ij} v_{k,i} v_{k,j} - W(\dot{\tau}) \right\} dV_0 \\ - \int_{S_F^{(0)}} \dot{F}_i^* v_i dS^{(0)} - \int_{S_V^{(0)}} \dot{F}_i (v_i - v_i^*) dS^{(0)} \end{aligned} \tag{VII.163}$$

In (VII.163) $S_F^{(0)}$, $S_V^{(0)}$ are the undeformed areas over which the traction rates F_i^* and velocities v_i^* are prescribed. The potential $W(\dot{\tau})$ is defined such that the time rate of change of Green's strain tensor E_{ij} is

$$\frac{\partial}{\partial t}E_{ij} \equiv \dot{E}_{ij} = \frac{\partial W}{\partial P_{ij}} \tag{VII.164}$$

The nominal surface tractions per unit undeformed area F_i are given by

$$F_j = (P_{ij} + P_{ik}u_{j,k})n_i^{(0)} \tag{VII.165}$$

with $\boldsymbol{n}^{(0)}$ the unit normal vector to the undeformed area. Using the simplifications indicated in [52], Williams takes

$$\begin{aligned} \dot{P}_{ij}\dot{E}_{ij} &= \dot{P}_{11}\dot{E}_{11} + 2\dot{P}_{12}\dot{E}_{12} + \dot{P}_{22}\dot{E}_{22} \\ P_{ij}v_{k,i}v_{k,j} &= P_{11}\dot{w}_1^2 + 2P_{12}\dot{w}_1\dot{w}_2 + P_{22}\dot{w}_2^2 \end{aligned} \tag{VII.166}$$

with

$$E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{3,i}u_{3,j}) \tag{VII.167}$$

The effect of a temperature variation is accounted for in Williams [51] by adding the term $\alpha\dot{T}(\dot{P}_{11} + \dot{P}_{22})$ to Nealeś [52] form for $W(\dot{\tau})$, i.e.,

$$\begin{aligned} W(\dot{\tau}) = \frac{1}{2}(c_{11}\dot{P}_{11}^2 + c_{22}\dot{P}_{22}^2) + c_{12}\dot{P}_{11}\dot{P}_{22} + c_{13}\dot{P}_{11}\dot{P}_{12} \\ + c_{23}\dot{P}_{22}\dot{P}_{12} + c_{33}\dot{P}_{12}^2 + \alpha\dot{T}(\dot{P}_{11} + \dot{P}_{22}) \end{aligned} \tag{VII.168}$$

Following the analysis in Neale [52] the material coefficients c_{ij} in [51] have the form

$$\begin{aligned} c_{11} &= \frac{1}{E} + \frac{G(2P_{11} - P_{22})^2}{9}, \quad c_{22} = \frac{1}{E} + \frac{G(2P_{22} - P_{11})^2}{9} \\ c_{12} &= -\frac{\nu}{E} + \frac{G(2P_{11} - P_{22})(2P_{22} - P_{11})}{9}, \quad c_{13} = \frac{2GP_{12}(2P_{11} - P_{22})}{3} \\ c_{23} &= \frac{2GP_{12}(2P_{22} - P_{11})}{3}, \quad c_{33} = \frac{1+\nu}{E} + 2GP_{12}^2 \end{aligned} \tag{VII.169}$$

In (VII.169) the hardening parameter G is identified using uniaxial stress-strain data and the basic assumption that plastic behavior depends on the stress only through the invariant

J_2; following Neale [52], G has the form

$$G = \frac{3}{4J_2}\left(\frac{1}{E_t} - \frac{1}{E}\right) \tag{VII.170}$$

where $\frac{1}{E_t}$, $E_t = E_t(J_2)$, is the slope of the graph of E_{11} vs. P_{11} at $T =$ const. Introducing the Ramberg-Osgood form

$$E_{11} - \alpha\delta T = \frac{1}{E}P_{11} + \left(\frac{P_{11}}{E_0}\right)^k \tag{VII.171}$$

where $\delta T = T - T_0$, T_0 the reference temperature, it follows that

$$G = \frac{3k(3J_2)^{(k-1)/2}}{4J_2E_0^k} \tag{VII.172}$$

so that the choice $k = 3$ leads to $G = \frac{27}{4}/E_0^3$; this restriction to $k = 3$ is then followed in the remainder of the development in [51]. Assuming that the effects of bending and stretching are equally important in inducing stress in an elastic-plastic plate, that spatial rates of change are significant over distances of order a, and introducing the associated order-of-magnitude estimates

$$\left\{\begin{aligned} M_{ij} &= hN_{ij}\mathcal{O}(1) \\ (u, v, w) &= h\left(\frac{h}{a}, \frac{h}{a}, 1\right)\mathcal{O}(1) \end{aligned}\right. \tag{VII.173}$$

it may be shown that

$$\begin{aligned} \int_{-h/2}^{h/2} \dot{F}_j v_j dz\Big|_{x=\pm a} = \pm\Big[&\dot{N}_{11}\dot{v} + \dot{N}_{12}\dot{v} - \dot{M}_{11}\dot{w}_1 - \dot{M}_{12}\dot{w}_2 \\ &+\dot{w}\frac{\partial}{\partial t}(Q_1 + N_{11}w_1 + N_{12}w_2)\Big] \end{aligned} \tag{VII.174a}$$

$$\begin{aligned} \int_{-h/2}^{h/2} \dot{F}_j v_j dz\Big|_{y=\pm b} = \pm\Big[&\dot{N}_{12}\dot{u} + \dot{N}_{22}\dot{v} - \dot{M}_{12}\dot{w}_1 - \dot{M}_{22}\dot{w}_2 \\ &+\dot{w}\frac{\partial}{\partial t}(Q_2 + N_{12}w_2 + N_{22}w_2)\Big] \end{aligned} \tag{VII.174b}$$

where

$$(Q_1, Q_2) = \int_{-h/2}^{h/2} (P_{13}, P_{23}) dz \tag{VII.174}$$

In writing down (VII.174a) and (VII.174b) terms of order $(h/a)^2$ have been neglected. Integrating the bending moment M_{12} by parts we obtain

$$\begin{aligned}\int_S \dot{F}_i v_i dS = -2[\dot{M}_{12}\dot{w}]\Big|_{-a}^{a}\Big|_{-b}^{b} + \int_{-b}^{b} [\dot{N}_{11}\dot{u} + \dot{N}_{12}\dot{v} + \dot{V}_1\dot{w} - \dot{M}_{11}\dot{w}_1]\Big|_{-a}^{a} dy \\ + \int_{-a}^{a} [\dot{N}_{12}\dot{u} + \dot{N}_{22}\dot{v} + \dot{V}_2\dot{w} - \dot{M}_{22}\dot{w}_2]\Big|_{-b}^{b} dx\end{aligned} \tag{VII.175}$$

where the net shear terms V_1, V_2 are given by

$$\begin{aligned} V_1 &= Q_1 + M_{12,2} + N_{11}w_1 + N_{12}w_2 \\ V_2 &= Q_2 + M_{12,1} + N_{12}w_1 + N_{22}w_2 \end{aligned} \tag{VII.176}$$

By carrying out the calculation to obtain the Euler equations from the variational equation $\delta\mathcal{I}^{(0)} = 0$, the following results are obtained in [51]:

$$\dot{N}_{\alpha\beta,\beta} = 0 \tag{VII.177}$$

$$\frac{\partial}{\partial t}\{M_{\alpha\beta,\alpha\beta} + (N_{\alpha\beta}w_\beta)_{,\alpha}\} = 0 \tag{VII.178}$$

and

$$\begin{bmatrix} \dot{\epsilon}_{11}^{(0)} - \dfrac{\dot{N}_T + \dot{N}_{11} - \nu\dot{N}_{22}}{Eh} \\ \dot{\epsilon}_{22}^{(0)} - \dfrac{\dot{N}_T + \dot{N}_{22} - \nu\dot{N}_{11}}{Eh} \\ 2\dot{\epsilon}_{12}^{(0)} - \dfrac{2(1+\nu)\dot{N}_{12}}{Eh} \\ -\dot{w}_{11} - \dfrac{12(\dot{M}_T + \dot{M}_{11} - \nu\dot{M}_{22})}{Eh^3} \\ -\dot{w}_{22} - \dfrac{12(\dot{M}_T + \dot{M}_{22} - \nu\dot{M}_{11})}{Eh^3} \\ -2\dot{w}_{12} - \dfrac{24(1+\nu)\dot{M}_{12}}{Eh^3} \end{bmatrix} = \frac{G}{9h^3} \cdot \boldsymbol{G} \begin{bmatrix} \dot{N}_{11} \\ \dot{N}_{22} \\ \dot{N}_{12} \\ \dot{M}_{11} \\ \dot{M}_{22} \\ \dot{M}_{12} \end{bmatrix} \tag{VII.179}$$

where $N_T,\ M_T$ are given by the usual relations, i.e.,

$$(N_T, M_T) = \alpha E \int_{-h/2}^{h/2} \delta T(1, z)dz$$

and where the hardening matrix $\boldsymbol{G}$ is defined in the appendix to this section ($\boldsymbol{G}$ is symmetric). If we let a superposed asterisk, again, denote quantities which are prescribed along the boundary of the plate then the natural boundary conditions assume the following form:

$$\begin{aligned}
&\dot{N}_{11} = \dot{N}_{11}^* \quad \text{or} \quad \dot{u} = \dot{u}^* \\
&\dot{N}_{12} = \dot{N}_{12}^* \quad \text{or} \quad \dot{v} = \dot{v}^* \\
&\dot{M}_{11,1} + 2\dot{M}_{12,2} + \frac{\partial}{\partial t}(N_{11}w_1 + N_{12}w_2) = \dot{V}_1^* \quad \text{or} \quad \dot{w} = \dot{w}^* \\
&\dot{M}_{11} = \dot{M}_{11}^* \quad \text{or} \quad \dot{w}_1 = \dot{w}_1^* \\
&\text{on } x = \pm a \text{ and} \\
&\dot{N}_{12} = \dot{N}_{12}^* \quad \text{or} \quad \dot{u} = \dot{u}^* \\
&\dot{N}_{22} = \dot{N}_{22}^* \quad \text{or} \quad \dot{v} = \dot{v}^* \\
&\dot{M}_{22,2} + 2\dot{M}_{12,1} + \frac{\partial}{\partial t}(N_{12}w_1 + N_{22}w_2) = \dot{V}^* \quad \text{or} \quad \dot{w} = \dot{w}^* \\
&\dot{M}_{22} = \dot{M}_{22}^* \quad \text{or} \quad \dot{w}_2 = \dot{w}_2^*
\end{aligned} \tag{VII.180}$$

at the corners of the plate. Taking

$$\begin{cases} V_1 = M_{11,1} + 2M_{12,2} + N_{11}w_1 + N_{12}w_2 \\ V_2 = M_{22,2} + 2M_{12,1} + N_{12}w_1 + N_{22}w_2 \end{cases} \tag{VII.181}$$

we find that

$$\begin{cases} \dot{Q}_1 = M_{11,1} + \dot{M}_{12,2} \\ \dot{Q}_2 = \dot{M}_{12,1} + \dot{M}_{22,2} \end{cases} \tag{VII.182}$$

and, thus obtain the form which is familiar from classical plate theory, i.e.,

$$\dot{M}_{\alpha\beta,\alpha\beta} = \dot{Q}_{\alpha,\alpha} \tag{VII.183}$$

Therefore, although the Neale [52] constitutive formulation differs from earlier formulations of the elastic-plastic deformation problem for plates, one does recover (in rate form) the equilibrium equations which are familiar from classical von Karman plate theory.

We now proceed with the derivation of the equations governing the buckling and post-buckling behavior of heated rectangular plates, assuming that the plate is constrained against displacement along its edges; such a plate, if elastic, will in general buckle if its edges are clamped or, if $M_T = 0$, if its edges are simply supported. In [51], attention is restricted to the case in which $M_T = 0$ and $N_T =$ const. over the surface of the plate (although, in principle, any spatial variation in N_T could be studied); under these restrictions it is easily shown that there exists a prebuckled membrane state given by

$$\begin{cases} u = v = w = 0 \\ N_{11} = N_{22} = \mathcal{N}(N_T) \\ N_{12} = 0 \\ M_{11} = M_{22} = M_{12} = 0 \end{cases} \tag{VII.184}$$

where

$$\dot{N}_T + \dot{\mathcal{N}}\left(1 - \nu + \frac{2EG\mathcal{N}^2}{9h^2}\right) = 0 \tag{VII.185}$$

is a consequence of the constitutive relations. It may be expected that this prebuckled membrane state persists for $N_T < (N_T)_{cr}$ with $(N_T)_{cr}$ corresponding to the critical temperature parameter.

In order to determine $(N_T)_{cr}$, and the behavior of the plate in a neighborhood of the critical temperature, Williams [51] adopts the procedure given by Danielson [63], i.e., the time variable is replaced by the distance parameter s along the load path measured from the critical point. With $\frac{\partial}{\partial t} = \frac{\partial}{\partial s}\frac{ds}{dt}$ we then assume the expansions

$$\begin{aligned} u &= su^{(1)} + s^2u^{(2)} + \ldots \\ v &= sv^{(1)} + s^2v^{(2)} + \ldots \\ w &= sw^{(1)} + s^2w^{(2)} + \ldots \end{aligned} \tag{VII.186}$$

and

$$
\begin{aligned}
N_{\alpha\beta} &= N\delta_{\alpha\beta} + sN^{(1)}_{\alpha\beta} + s^2 N^{(2)}_{\alpha\beta} + \ldots \\
M_{\alpha\beta} &= sM^{(1)}_{\alpha\beta} + s^2 M^{(2)}_{\alpha\beta} + \ldots \\
N &= N_{cr} + N^{(1)} + s^2 N^{(2)} + \ldots \\
N_T &= (N_T)_{cr} + sN^{(1)}_T + s^2 N^{(2)}_T + \ldots
\end{aligned}
\tag{VII.187}
$$

and obtain from the Euler (equilibrium) equations (VII.177)-(VII.178)

$$
\begin{cases}
N^{(1)}_{\alpha\beta,\beta} = 0 \quad N^{(2)}_{\alpha\beta,\beta} = 0 \\
M^{(1)}_{\alpha\beta,\alpha\beta} + N_{cr}\delta_{\alpha\beta} w^{(1)}_{\alpha\beta} = 0 \\
M^{(2)}_{\alpha\beta,\alpha\beta} + N_{cr}\delta_{\alpha\beta} w^{(2)}_{\alpha\beta} + (N^{(1)}_{\alpha\beta} + N^{(1)}\delta_{\alpha\beta}) w^{(1)}_{\alpha\beta} = 0
\end{cases}
\tag{VII.188}
$$

Also, from (VII.185) it follows that

$$
\begin{cases}
(N_T)_{cr} = -N_{cr}\left(1 - \nu + \dfrac{2g^*}{3}\right), \quad g^* = \dfrac{GEN^2_{cr}}{9h^2} \\
N^{(1)}_T = -N^{(1)}(1 - \nu + 2g^*) \\
N^{(2)}_T = -N^{(2)}(1 - \nu + 2g^*) - \dfrac{2g^* N^{(1)2}}{N_{cr}}
\end{cases}
\tag{VII.189}
$$

The Euler constitutive relations (VII.179) are now rewritten in the form

$$
\begin{aligned}
Eh\dot{\epsilon}^{(0)}_{\alpha\beta} &= \dot{N}_T \delta_{\alpha\beta} + A_{\alpha\beta\gamma\delta}\dot{N}_{\gamma\delta} + B_{\alpha\beta\gamma\delta}\dot{M}_{\gamma\delta} \\
-Eh\dot{w}_{\alpha\beta} &= \frac{12\dot{M}_T}{h^2}\delta_{\alpha\beta} + B_{\alpha\beta\gamma\delta}\dot{N}_{\gamma\delta} + D_{\alpha\beta\gamma\delta}\dot{M}_{\gamma\delta}
\end{aligned}
\tag{VII.190}
$$

where

$$
\begin{aligned}
&A_{1111} = A_{11}, \quad A_{1122} = A_{12}, \quad A_{1112} = A_{13}, \quad A_{1131} = A_{14} \\
&A_{2211} = A_{21}, \quad A_{2222} = A_{22}, \quad A_{2212} = A_{23}, \quad A_{2221} = A_{24} \\
&A_{1211} = A_{31}, \quad A_{1222} = A_{32}, \quad A_{1212} = A_{33}, \quad A_{1221} = A_{34} \\
&A_{2111} = A_{41}, \quad A_{2122} = A_{42}, \quad A_{2112} = A_{43}, \quad A_{2121} = A_{44}
\end{aligned}
\tag{VII.191}
$$

and the components of the symmetric $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{D}$ matrices are defined in the appendix to this section.

The strain tensor $\epsilon^{(0)}_{\alpha\beta}$ is decomposed according to

$$\epsilon^{(0)}_{\alpha\beta} = e_{\alpha\beta} + \frac{w_{,\alpha} w_{,\beta}}{2} \tag{VII.192}$$

and, furthermore, we write

$$\begin{cases} e_{\alpha\beta} = s e^{(1)}_{\alpha\beta} + s^2 e^{(2)}_{\alpha\beta} + \dots. \\ \kappa_{\alpha\beta} \equiv w_{,\alpha\beta} = s\kappa^{(1)}_{\alpha\beta} + s^2 \kappa^{(2)}_{\alpha\beta} + \dots \end{cases} \tag{VII.193}$$

where

$$\begin{cases} e^{(k)}_{11} = u^{(k)}_{,1}, \; e^{(k)}_{22} = v^{(k)}_{,2} \\ \\ e^{(k)}_{12} = e^{(k)}_{21} = \frac{1}{2}\left(u^{(k)}_{,2} + v^{(k)}_{,1}\right) \end{cases} \tag{VII.194}$$

The tensor components $A_{\alpha\beta\gamma\delta}$, $B_{\alpha\beta\gamma\delta}$, $D_{\alpha\beta\gamma\delta}$ are also expanded as series in the parameter s, i.e.,

$$\begin{aligned} A_{\alpha\beta\gamma\delta} \;\; &= \;\; A^{(0)}_{\alpha\beta\gamma\delta} + sA^{(1)}_{\alpha\beta\gamma\delta} + s^2 A^{(2)}_{\alpha\beta\gamma\delta} + \dots \\ & \;\;\;\;\;\; B_{\alpha\beta\gamma\delta} = sB^{(1)}_{\alpha\beta\gamma\delta} + s^2 B^{(2)}_{\alpha\beta\gamma\delta} + \dots \\ & \;\;\;\;\;\; D_{\alpha\beta\gamma\delta} = D^{(0)}_{\alpha\beta\gamma\delta} + sD^{(1)}_{\alpha\beta\gamma\delta} + s^2 D^{(2)}_{\alpha\beta\gamma\delta} + \dots \end{aligned} \tag{VII.195}$$

Using the expansions (VII.193), (VII.195) in the constitutive relations (VII.190), (VII.191) it follows that

$$\begin{aligned} &Ehe^{(1)}_{\alpha\beta} = N^{(1)}_T \delta_{\alpha\beta} + A^{(0)}_{\alpha\beta\gamma\delta}(N^{(1)}_{\gamma\beta} + N^{(1)}\delta_{\gamma\delta}) \\ &Eh\left(e^{(2)}_{\alpha\beta} + \frac{w^{(1)}_a w^{(1)}_\beta}{2}\right) = N^{(2)}_T \delta_{\alpha\beta} + A^{(0)}_{\alpha\beta\gamma\delta}(N^{(2)}_{\gamma\delta} + N^{(2)}\delta_{\gamma\delta}) \\ &\qquad + \frac{1}{2}A^{(1)}_{\alpha\beta\gamma\delta}(N^{(1)}_{\gamma\delta} + N^{(1)}\delta_{\gamma\delta}) + \frac{1}{2}B^{(1)}_{\alpha\beta\gamma\delta}M^{(1)}_{\gamma\delta} \\ &-Eh\kappa^{(1)}_{\alpha\beta} = D^{(0)}_{\alpha\beta\gamma\delta}M^{(1)}_{\gamma\delta} \\ &-Eh\kappa^{(2)}_{\alpha\beta} = D^{(0)}_{\alpha\beta\gamma\delta}M^{(2)}_{\gamma\delta} + \frac{1}{2}D^{(1)}_{\alpha\beta\gamma\delta}M^{(1)}_{\gamma\delta} + \frac{1}{2}B^{(1)}_{\alpha\beta\gamma\delta}(N^{(1)}_{\gamma\delta} + N^{(1)}\delta_{\gamma\delta}) \\ &-Eh\kappa^{(3)}_{\alpha\beta} = D^{(0)}_{\alpha\beta\gamma\delta}M^{(3)}_{\gamma\delta} + \frac{2}{3}D^{(1)}_{\alpha\beta\gamma\delta}M^{(2)}_{\gamma\delta} + \frac{1}{3}D^{(2)}_{\alpha\beta\gamma\delta}M^{(1)}_{\gamma\delta} \\ &\qquad + \frac{2}{3}B^{(1)}_{\alpha\beta\gamma\delta}(N^{(2)}_{\gamma\delta} + N^{(2)}\delta_{\gamma\delta}) + \frac{1}{3}B^{(2)}_{\alpha\beta\gamma\delta}(N^{(1)}_{\gamma\delta} + N^{(1)}\delta_{\gamma\delta}) \end{aligned} \tag{VII.196}$$

In (VII.196) one has a hierarchy of equations which must be solved to obtain the expansion variables $w^{(k)}$, etc. We note, however, that for displacements and stresses which satisfy the boundary conditions, identically, the variational functional $\mathcal{I}^{(0}$ may be simplified so as to read

$$\mathcal{I}^{(0)} = \int_{v_0} [\dot{P}_{ij}\dot{E}_{ij} + \frac{1}{2}P_{ij}v_{k,i}v_{k,j} - W(\dot{\tau})]dV_0 \tag{VII.197}$$

Using the order of magnitude estimates (VII.172), and restricting attention to stress rates which satisfy the constitutive relations, the variational equation $\delta\mathcal{I}^{(0)} = 0$ simplifies to

$$\int_{-a}^{a}\int_{-b}^{b} \{N_{\alpha\beta}(\delta\dot{e}_{\alpha\beta} + w_{,\alpha}\delta\dot{w}_{,\beta}) + N_{\alpha\beta}\dot{w}_{,\alpha}\delta\dot{w}_{,\beta} - M_{\alpha\beta}\delta\dot{\kappa}_{\alpha\beta}\}\, dxdy = 0 \tag{VII.198}$$

The prebuckled state given by $(M_T = 0)$

$$w = 0,\ N_{\alpha\beta} = \mathcal{N}(N_T)\delta_{\alpha\beta},\ M_{\alpha\beta} = 0 \tag{VII.199}$$

must satisfy (VII.198), i.e.,

$$\int_{-a}^{a}\int_{-b}^{b} \dot{\mathcal{N}}(N_T)\delta_{\alpha\beta}\delta\dot{e}_{\alpha\beta}dxdy = 0 \tag{VII.200}$$

We now substitute the expansions (VII.186), (VII.187) into both (VII.198) and (VII.200) and subtract the resulting equations so as to obtain

$$\begin{aligned}\int_{-a}^{a}\int_{-b}^{b} & (N^{(1)}_{\alpha\beta}\delta\dot{e}_{\alpha\beta} + N^{*}w^{(1)}_{\alpha}\delta_{\alpha\beta}\delta\dot{w}_{\beta} - M^{(1)}_{\alpha\beta}\delta\dot{\kappa}_{\alpha\beta} \\ & +2s\{N^{(2)}_{\alpha\beta}\delta\dot{e}_{\alpha\beta} + (w^{(1)}_{\alpha}(N^{(1)}_{\alpha\beta} + N^{(1)}\delta_{\alpha\beta}) + N^{*}w^{(2)}_{\alpha}\delta_{\alpha\beta})\delta\dot{w}_{\beta} \\ & -M^{(2)}_{\alpha\beta}\delta\dot{\kappa}_{\alpha\beta} + 3s^2N^{(3)}_{\alpha\beta}\delta\dot{e}_{\alpha\beta} + [w^{(2)}_{\alpha}(N^{(1)}_{\alpha\beta} + N^{(1)}\delta_{\alpha\beta} \\ & +w^{(1)}_{\alpha}(N^{(2)}_{\alpha\beta} + N^{(2)}\delta_{\alpha\beta}) + N^{*}w^{(3)}_{\alpha}\delta_{\alpha\beta}]\delta\dot{w}_{\beta} - M^{(3)}_{\alpha\beta}\delta\dot{\kappa}_{\alpha\beta}| + \ldots)dxdy = 0\end{aligned} \tag{VII.201}$$

Restricting the strain-rate variations to have the form

$$\delta\dot{e}_{\alpha\beta} = se^{(k)}_{\alpha\beta},\ \delta\dot{w}_{,\beta} = \dot{s}w^{(k)}_{,\beta},\ \delta\dot{r}_{\alpha\beta} = \dot{s}\kappa^{(k)}_{\alpha\beta} \tag{VII.202}$$

equation (VII.201) becomes, as $s \to 0$,

$$\int_{-a}^{a}\int_{-b}^{b} (N^{(1)}_{\alpha\beta}e^{(k)}_{\alpha\beta} + \mathcal{N}w^{(1)}_{,\alpha}\delta_{\alpha\beta}w^{(k)}_{,\beta} - M^{(k)}_{\alpha\beta})dxdy = 0 \tag{VII.203}$$

To avoid the possibility that $w^{(k)}$ contains an arbitrary multiple of $w^{(k)}$, Williams [51] imposes the restriction

$$\int_{-a}^{a}\int_{-b}^{b}\delta_{\alpha\beta}w^{(k)}_{,\alpha}w^{(k)}_{,\beta}\,dxdy = 0 \tag{VII.204}$$

the consequence of which is

$$\int_{-a}^{a}\int_{-b}^{b}(N^{(1)}_{\alpha\beta}e^{(k)}_{\alpha\beta} - M^{(1)}_{\alpha\beta}\kappa^{(k)}_{\alpha\beta})dxdy = 0 \tag{VII.205}$$

On the other hand, if the strain-rate variations are restricted to have the form

$$\delta\dot{e}_{\alpha\beta} = \dot{s}e^{(1)}_{\alpha\beta},\ \delta\dot{w}_{,\beta} = \dot{s}w^{(1)}_{,\beta},\ \delta\dot{\kappa}_{\alpha\beta} = \dot{s}\kappa^{(1)}_{\alpha\beta} \tag{VII.206}$$

then (VII.201) becomes, with the aid of (VII.203)-(VII.205),

$$\begin{aligned}&\left(N^{(1)} + \frac{3sN^{(2)}}{2} + \dots\right)\int_{-a}^{a}\int_{-b}^{b}(\delta_{\alpha\beta}w^{(1)}_{\alpha}w^{(1)}_{\beta}\\ &+(N^{(2)}_{\alpha\beta}e^{(1)}_{\alpha\beta} - N^{(1)}_{\alpha\beta}e^{(2)}_{\alpha\beta}) + N^{(1)}_{\alpha\beta}w^{(1)}_{\alpha}w^{(1)}_{\beta} - (M^{(2)}_{\alpha\beta}K^{(1)}_{\alpha\beta} - M^{(2)}_{\alpha\beta}K^{(2)}_{\alpha\beta})\\ &+\frac{3s}{2}[N^{(3)}_{\alpha\beta}e^{(1)}_{\alpha\beta} - N^{(1)}_{\alpha\beta}e^{(3)}_{\alpha\beta} + N^{(1)}_{\alpha\beta}w^{(2)}_{\alpha}w^{(1)}_{\beta} + N^{(2)}_{\alpha\beta}w^{(1)}_{\alpha}w^{(1)}_{\beta}\\ &(M^{(3)}_{\alpha\beta}K^{(1)}_{\alpha\beta} - M^{(1)}_{\alpha\beta}K^{(3)}_{\alpha\beta})]dxdy = 0\end{aligned} \tag{VII.207}$$

By setting the coefficients of s^0 and s^1 equal to zero in (VII.207), the equations for $N^{(1)}$ and $N^{(2)}$ are obtained; in fact, by anticipating that $e^{(1)}_{\alpha\beta}$ and $N^{(1)}_{\alpha\beta}$ vanish it follows from (VII.207) that

$$\begin{aligned}N^{(1)}&\int_{-a}^{a}\int_{-b}^{b}\delta_{\alpha\beta}w^{(1)}_{,\alpha}w^{(1)}_{,\beta}\,dxdy\\ &= \int_{-a}^{a}\int_{-b}^{b}(M^{(2)}_{\alpha\beta}\kappa^{(2)}_{\alpha\beta} - M^{(1)}_{\alpha\beta}\kappa^{(2)}_{\alpha\beta})dxdy\end{aligned} \tag{VII.208}$$

It is shown, in [51], that as a consequence of (VII.208) $N^{(1)} = 0$, in which case it also follows that $N^{(1)}_T = 0$. From (VII.207) one then obtains

$$\begin{aligned}&N^{(2)}\int_{-a}^{a}\int_{-b}^{b}\delta_{\alpha\beta}w^{(1)}_{\alpha}w^{(1)}_{\beta}\,dxdy + \int_{-a}^{a}\int_{-b}^{b}\{N^{(2)}_{\alpha\beta}w^{(1)}_{\alpha}w^{(1)}_{\beta}\\ &\frac{M^{(1)}_{\alpha\beta}}{3Eh}[D^{(2)}_{\alpha\beta\gamma\delta}M^{(1)}_{\gamma\delta} + 2B^{(1)}_{\alpha\beta\gamma\delta}(N^{(2)}_{\gamma\delta}\delta_{\gamma\delta})]\} = 0\end{aligned} \tag{VII.209}$$

and, thus, the curvature term $N^{(2)}$, and hence $N_T^{(2)}$ can be computed without having to use the third-order equations.

The critical temperature $(N_T)_{cr}$ at which the rectangular plate (which is heated so that $N_T = \text{const.}$ and $M_T = 0$) will buckle may now be determined along with the form of the dependence of the buckling temperature on the hardening parameter G; such determinations are limited in [51] to those cases for which an analytical solution exists, i.e., the fully simply supported plate, the clamped-simply supported plate, and the long strip. In all cases, we assume that the in-plane displacements vanish on the boundaries.

We begin by noting that the first pair of constitutive relatinos in (VII.196) may be rewritten in the form

$$\begin{aligned} N_{11}^{(1)} &= \frac{Eh/(1+\nu)}{1-\nu+2g^*}\left[(1+g^*)\frac{\partial u^{(1)}}{\partial x} + (\nu - g^*)\frac{\partial v^{(1)}}{\partial y}\right] \\ N_{22}^{(1)} &= \frac{Eh/(1+\nu)}{1-\nu+2g^*}\left[(\nu - g^*)\frac{\partial u^{(1)}}{\partial x} + (1+g^*)\frac{\partial v^{(1)}}{\partial y}\right] \\ N_{12}^{(1)} &= \frac{Eh/2}{1+\nu}\left(\frac{\partial u^{(1)}}{\partial y} + \frac{\partial v^{(1)}}{\partial x}\right) \end{aligned} \tag{VII.210}$$

and if these relations are then substituted into the in-plane equations of equilibrium it can be shown that both $u^{(1)}$ and $v^{(1)}$ satisfy the biharmonic equation; in conjunction with homogeneous boundary data it may then be concluded that $u^{(1)}, v^{(1)}$, and, hence, $e_{\alpha\beta}^{(1)}$ in general vanish. As a consequence of (VII.210) it then follows that the $N_{\alpha\beta}^{(1)}$ also vanish; thus, the conditions hold which validate (VII.208) in which case $N^{(1)} = 0$. Using the equation governing transverse equilibrium, and the remaining constitutive relations in (VII.196), it follows that $w^{(1)}$ is the nonzero solution of

$$\nabla^2\left(\nabla^2 w^{(1)} - \frac{12}{a^2}\bar{N}_{cr}\left\{\frac{1-\nu+2g^*}{1+g^*}\right\}w^{(1)}\right) = 0 \tag{VII.211}$$

where

$$
\begin{aligned}
M_{11}^{(1)} &= \frac{-Eh^3/[12(1+\nu)]}{1-\nu+2g^*}[(1+g^*)w_{11}^{(1)} + (1+g^*)w_{22}^{(1)}] \\
M_{22}^{(1)} &= \frac{-Eh^3/[12(1+\nu)]}{1-\nu+2g^*}[(\nu-g^*)w_{11}^{(1)} + (1+g^*)w_{22}^{(1)}] \\
M_{12}^{(1)} &= \frac{-Eh^3/12}{1+\nu}w_{12}^{(1)}
\end{aligned}
\tag{VII.212}
$$

and we have defined

$$
\begin{cases}
(N_{cr}, (N_T)_{cr}) = \dfrac{Eh^3/a^3}{1+\nu}(\bar{N}_{cr}, (\bar{N}_T)_{cr}) \\
g^* = \bar{G}\left(\dfrac{\bar{N}_{cr}}{1+\nu}\right)^2 \\
\bar{G} = \dfrac{1}{9}EG\left(\dfrac{Eh^2}{a^2}\right)^2
\end{cases}
\tag{VII.213}
$$

We note that there exists an infinite number of eigenvalues λ_n of (VII.211), subject to appropriate boundary conditions, such that

$$
\lambda_n^2 = -12\bar{N}_{cr}\left\{\frac{1-\nu+g^*}{1+g^*}\right\}. \tag{VII.214}
$$

We also note that as g^* depends on N_{cr} though (VII.213), N_{cr} is a solution of the cubic equation

$$
\bar{N}_{cr}^3 + \frac{\lambda_n^2}{24}\bar{N}_{cr}^2 + \frac{(1+\nu)^2}{2\bar{G}}\left\{(1-\nu)\bar{N}_{cr} + \frac{\lambda_n^2}{12}\right\} = 0 \tag{VII.215}
$$

Thus the minimum buckling stress $\bar{N}_{cr}$ can be determined only after one examines the roots of (VII.215) which correspond to the previously determined eigenvalues λ_n. As Williams [51] notes, the limiting case $\bar{G}=0$ yields some insight into the dependence $(\bar{N}_T)_{cr}(G)$; if we consider λ_n as being known, then (VII.215) determines the functional relationship $\bar{N}_{cr}(G)$. Also, as

$$
(\bar{N}_{cr})\Big|_{\bar{G}=0} = -\frac{\lambda_n^2}{12(1-\nu)} \quad \text{and} \quad \frac{dg^*}{d\bar{G}}\Big|_{\bar{G}=0} = \left(\frac{\bar{N}_{cr}(0)}{1+\nu}\right)^2 \tag{VII.216}
$$

it follows, as a consequence of (VII.215), that

$$
\frac{d(\bar{N}_T)_{cr}}{d\bar{G}}\Big|_{\bar{G}=0} = \frac{1+3\nu}{(1+\nu)^2}\cdot\frac{(\bar{N}_{cr})^3(0)}{3} \tag{VII.217}
$$

As $(\bar{N}_{cr})(0) < 0$ the effect of the hardening parameter is always to reduce the buckling temperature and the absolute value of the stress from those values which apply for a linearly elastic material in a small neighborhood of $G = 0$.

The solution of (VII.211) which is the easiest to secure is the one which corresponds to the case in which the plate is simply supported along all its edges.

Taking

$$w^{(1)} = A_{mn} \sin\left[\frac{n\pi}{2a}(x+a)\right] \sin\left[\frac{m\pi}{2b}(y+b)\right] \tag{VII.218}$$

it follows that the boundary conditions along the edges $x = \pm a$, $y = \pm b$ are identically satisfied and that the eigenvalues are

$$\lambda_{mn}^2 = \frac{\pi^2}{4}(n^2 + m^2\phi^2);\ \phi = a/b \tag{VII.219}$$

For $\nu = 0.3$, numerical results for the buckling stress and temperatures are reported by Williams in [51] and are summarized in tables 4 and 5. For all the cases shown the minimum values of stress and temperature correspond to $m = n = 1$, i.e., a mode shape with one wave in each direction. The effect of the hardening parameter is to reduce the stress and temperature from values which would correspond to $G = 0$; this effect is more pronounced, as noted in [51], for long slender plates. Also the effect of G is not monotonic: the buckling temperature first decreases with increasing G and then increases.

For the case in which the edges at $x = \pm a$ are simply supported, while those at $y = \pm b$ are clamped, we may take

$$w^{(1)} = W(\bar{y}) \sin\left[\frac{n\pi}{2a}(x+a)\right];\ \bar{y} = \frac{y}{h} \tag{VII.220}$$

so that the boundary conditions along $x = \pm a$ are satisfied identically while $w(\bar{y})$ must satisfy

$$W'''' + (\beta^2 - \bar{n}^2)W'' - \bar{n}^2\beta^2 W = 0 \tag{VII.221}$$

with

$$\bar{n} = \frac{n\pi b}{2a};\ \bar{n}^2 + \beta^2 = \left(\frac{\lambda n}{\phi}\right)^2 \tag{VII.222}$$

The solution of (VII.221) has the form

$$W = A_1 \sin h\bar{n}\bar{y} + A_2 \cos h\bar{n}\bar{y} + A_3 \sin \beta\bar{y} + Ay \cos \beta\bar{y} \tag{VII.223}$$

From the clamped conditions along $\bar{y} = \pm 1$, we infer that either the buckling mode is even ($A_1 = A_3 = 0$) with β satisfying

$$\beta \tan \beta = -\bar{n} \tan h\bar{n} \tag{VII.224}$$

or the buckling mode is odd ($A_2 = A_4 = 0$) with

$$\frac{\tan \beta}{\beta} = \frac{\tan h\bar{n}}{\bar{n}} \tag{VII.225}$$

Therefore, in order to determine the eigenvalues λ_n, and then the buckling stress and temperature, (VII.224) or (VII.225) must be solved, for a fixed $\bar{n}$, for $\beta(\bar{n})$. Numerical results for this clamped-simply supported case are delineated in tables 6 and 7, for $\nu = 0.3$; in all the cases exhibited the minimum values of the buckling temperature and stress correspond to the even mode case ($A_1 = A_3 = 0$ in (VII.223)) and, thus, to the smallest root β of (VII.224). Once again, the effect of G is to, in general, reduce the buckling stress from those values associated with $G = 0$, but the reduction in the buckling temperature due to G appears to occur in higher-aspect ratio plates only and only in a small neighborhood of $G = 0$.

Williams [51] also treats the initial buckling of a long strip undergoing both cylindrical as well as non-cylindrical deformations; for the case of a strip with simply supported edges the deflection has the form

$$w^{(1)} = A_s \sin \frac{\lambda x}{a} + B_s \cos \frac{\lambda x}{a} \tag{VII.226}$$

where $2a$ is the width of the strip, while for a strip with clamped edges at $x = \pm a$

$$w^{(1)} = A_c \sin \frac{\lambda x}{a} + B_c \cos \frac{\lambda x}{a} + \frac{a^2}{\lambda^2}(c_1 x + c_2) \tag{VII.227}$$

In the simply supported case we have

$$\begin{array}{c} B_s \neq 0,\ A_s = 0;\ \lambda = \frac{\pi}{2},\ \frac{3\pi}{2}, \ldots \\ \text{or} \\ A_s \neq 0,\ B_s = 0,\ \lambda = \pi,\ 2\pi, \ldots \end{array} \tag{VII.228}$$

while in the clamped case we have

$$\begin{array}{c} B_c \neq 0,\ A_c = 0;\ \lambda = \pi,\ 2\pi, \ldots \\ \text{or} \\ A_c \neq 0,\ B_c = 0;\ \tan\lambda = \lambda(\lambda = 4.4934,\ 7.7252, \ldots) \end{array} \tag{VII.229}$$

Numerical results for these two situations are delineated in tables 8 and 9 with $\nu = 0.3$; for all the cases depicted, the minimum buckling stress and temperature are those which are associated with an even mode. The effect of G is to reduce both the buckling temperature and stress with the effect now monotonic over the range of G considered.

We now want to briefly summarize some of the results obtained in [51] relative to the postbuckling behavior of the plate and the nature of the dependence of that behavior on the hardening parameter G. The first parameter which arises in a postbuckling calculation is $w^{(2)}(s)$ which is the initial curvature of the transverse deflection as measured along the loading curve that initiates at the critical point. As $N^{(1)} = 0$, it follows from the equilibrium and constitutive equations that $w^{(2)}$ satisfies the same equation as $w^{(1)}$ and, thus, by virtue of the orthogonality relation (VII.204), $w^{(1)} = 0$. The next parameter which must be computed is $N^{(2)}$. For the specific case of cylindrical deformation of a long strip the second order constitutive equations assume the form

$$\begin{aligned} Eh(u_1^{(2)} + \frac{1}{2}w_1^{(1)2}) &= (1+g^*)N_{11}^{(2)} + (-\nu + g^*)N_{22}^{(2)} + \frac{6g^*(4M_{11}^{(1)2} - M_{11}^{(1)}M_{22}^{(1)} + M_{22}^{(1)2})}{N_{cr}h^2} \\ 0 &= (1+g^*)N_{22}^{(2)} + (-\nu + g^*)N_{11}^{(2)} + \frac{6g^*(4M_{22}^{(1)2} - M_{11}^{(1)}M_{22}^{(1)} + M_{11}^{(1)2})}{N_{cr}h^2} \end{aligned} \tag{VII.230}$$

However,

$$M_{22}^{(1)} = \frac{\nu - g^*}{1+g^*}M_{11}^{(1)},\ M_{11}^{(1)} = -\frac{Eh^3}{12(1+\nu)}\frac{1+g^*}{1-\nu+2g^*}w_{11}^{(1)} \tag{VII.231}$$

so that

$$\begin{aligned} Eh(1+g^*)(u_1^{(2)} + \frac{1}{2}w_1^{(1)2}) &= (1+\nu)(1-\nu+2g^*)N_{11}^{(2)} \\ &+ \frac{24(1+\nu)g^*}{N_{cr}h^2(1+g^*)^2}[(1-\nu)(1+3g^*)+\nu^2+3g^*]M_{11}^{(1)2} \end{aligned} \tag{VII.232}$$

while the second-order equilibrium equation (VII.188) requires that $N_{11}^{(2)} = \text{const}$. The form of $N_{11}^{(2)}$ can be obtained by integrating (VII.232) over $-a \leq x \leq a$ and using the fact that $u^{(2)}(\pm a) = 0$. In both the clamped and simply supported cases one obtains (with $B_i = B_s$ or B_c)

$$N_{11}^{(2)} = -\frac{3N_{cr}B_i^2}{h^2}\left[1 + \frac{4g^*}{(1+g^*)^2}\frac{(1-\nu)(1+3g^*)+\nu^2+3g^{*2}}{1-\nu+2g^*}\right] \tag{VII.233}$$

The form of $N^{(2)}$ may now be computed by employing (VII.209); numerical results for both $N^{(2)}$ and $N_{11}^{(2)}$ (with $\nu = 0.3$) are presented in tables 10 and 11; table 10 covers the case of a clamped strip while table 11 addresses the same situation for a simply supported strip. As

$$\frac{N_{11}}{N_{cr}} = 1 + \frac{s^2(N_{11}^{(2)} + N^{(2)})}{N_{cr}} + \mathcal{O}(s^3) \tag{VII.234}$$

and $N_{11}^{(2)} + N^{(2)} = \mathcal{O}(g^*) > 0$, the stress resultant N_{11} decreases in absolute value, after buckling, as a consequence of hardening; moreover, the effect of hardening is to increase the rate of decrease of N_{11}. We indicate in Figs. 44 and 45, respectively, some numerical results for the second-order stress parameter $N_{22}^{(2)}$ in the simply supported and clamped cases; in constrast with N_{11} we have $N_{22}^{(2)} + N^{(2)} < 0$ so that the stress resultant N_{22} decreases (actually increases in absolute value) after buckling in a manner which is independent of hardening. As noted in Williams [51], at $x = 0$, the centerline of the plate, the effect of hardening is to effect a small decrease in the rate of decrease of $N_{22}^{(2)}$.

We have presented, above, the equations used by William [51] to study the buckling of elastoplastic rectangular plates; the model chosen leads to a modified form of von Karman's equations in which the constitutive relations are generalized so as to account for the flow rule. It has been shown that, in for clamped plates and for simply supported plates in which there is no thermal moment, i.e., $M_T = 0$, buckling may occur at temperatures that are substantially below those which are predicted by an elastic analysis; in fact, as the effect of g^* is to reduce the factors which multiply the curvature terms in our equations, the elastoplastic equations essentially describe a classical plate with reduced stiffness. Thus, as Williams [51] explicitly notes, it is not suprising that the equations in question predict a general reduction in the temperature which is necessary to cause buckling. The predictions in [51] are in general (qualitative) agreement with the results given by Gerard and Becker [64] for mechanically loaded plates; some of these results have been described by the authors in [1]. For example, for long plates which are either clamped or simply supported, respectively, on the unloaded sides, the plasticity reduction factor η, which is the ratio of the critical buckling loads in the elastic and plastic cases, is given by

$$\eta = \frac{\sigma_{cr}^p}{\sigma_{cr}^e} = f\left[(0.352, 0.500) + (0.324, 0.250)\sqrt{1 + \frac{3E_t}{E_s}}\right] \tag{VII.235}$$

In (VII.235),

$$f = \frac{E_s(1-\nu_e^2)}{E(1-\nu^2)},\ \nu = \frac{1}{2} - \left(\frac{1}{2} - \nu_e\right)\frac{E_s}{E} \tag{VII.236}$$

where ν_e, E_s, and E_t are the elastic Poisson's ratio, the secant modulus, and the tangent modulus of the maternal, respectively. For the Ramberg-Osgood form (VII.170)

$$\begin{cases} \dfrac{E}{E_t} = 1 + \dfrac{3E}{E_0}\left(\dfrac{P_{11}}{E_0}\right)^2 \equiv 1 + \dfrac{9EJ_2}{E_0^3} \\ \dfrac{E}{E_s} = 1 + \dfrac{E}{E_0}\left(\dfrac{P_{11}}{E_0}\right)^2 \equiv 1 + \dfrac{3EJ_2}{E_0^3} \end{cases} \tag{VII.237}$$

where $E_t = E_t(J_2)$ and $J_2 = \frac{1}{3}P_{11}^2$. However, as $P_{11} = P_{22} = N_{cr}/h$ and $P_{12} = 0$, at the critical point, one obtains from (VII.237)

$$\frac{E}{E_t} = 1 + 4g^*,\ \frac{E}{E_s} = 1 + \frac{4g^*}{3} \tag{VII.238}$$

A comparison of the results in [64] and [51] is presented in table 12; in this table, the predictions associated with a long $\left(\phi = \frac{a}{b} = 3.5\right)$, (completely) simply supported plate appear in the first column while those in the second column correspond to the case of a clamped strip. As noted by Williams [51], the general behavior indicated in table 12 is the same for both sets of boundary conditions. Concerning the postbuckling analysis, it should be noted that the condition of loading ($\dot{J}_2 > 0$) is generally violated once buckling has occurred. In fact, to first order, the plane stress form for J_2 is given by

$$J_2 = \frac{1}{3}(P_{11}^2 + P_{22}^2 - P_{11}P_{22}) + P_{12}^2 \tag{VII.239}$$

Thus, as

$$\dot{J}_2 = \left\{ \frac{1}{3} \left[\frac{\partial P_{11}}{\partial s}(2P_{11} - P_{22}) + \frac{\partial P_{22}}{\partial s}(2P_{22} - P_{11}) \right] + 2P_{12}\frac{\partial P_{12}}{\partial s} \right\} \frac{ds}{dt}, \tag{VII.240}$$

$$\begin{cases} \dfrac{\partial P_{ij}}{\partial s} = \dfrac{12y}{h^3} M_{ij}^{(1)} + \mathcal{O}(s) \\ (2P_{11} - P_{22})(2P_{22} - P_{11}) = \dfrac{N_{cr}}{h} + \mathcal{O}(s), \end{cases} \tag{VII.241}$$

and

$$M_{11}^{(1)} + M_{22}^{(1)} = -\frac{Eh^3}{12} \left\{ \frac{w_{11}^{(1)} + w_{22}^{(1)}}{1 - \nu + 2g^*} \right\} \tag{VII.242}$$

it follows that

$$\dot{J}_2 = \left\{ -\frac{EyN_{cr}/3h}{1 - \nu + 2g^*}(w_{11}^{(1)} + w_{22}^{(1)}) + \mathcal{O}(s) \right\} \dot{s} \tag{VII.243}$$

If the first curvature is positive, i.e., $w_{11}^{(1)} + w_{22}^{(1)} > 0$ then, because $N_{cr} < 0$, there will be unloading on the $y < 0$ (negative) side of the plate; as indicated in [51] this result is consistent with the expectation that the bending moments due to buckling will induce tension in the domain $y < 0$ and, thus, produce unloading.

Appendix: The Hardening Matrix G and Related Matrices

The constitutive equations are initially expressed in terms of the symmetric G matrix, whose elements are defined as

$$G_{11} = (2N_{11} - N_{22})^2 + \frac{12(2M_{11} - M_{22})^2}{h^2}$$

$$G_{12} = (2N_{11} - N_{22})(2N_{22} - N_{11}) + \frac{12(2M_{11} - M_{22})(2M_{22} - M_{11})}{h^2}$$

$$G_{13} = 6\left[N_{12}(2N_{11} - N_{22}) + \frac{12M_{12}(2M_{11} - M_{22})}{h^2}\right]$$

$$G_{14} = \frac{24(2N_{11} - N_{22})(2M_{11} - M_{22})}{h^2}$$

$$G_{15} = \frac{12[5(N_{11}M_{22} + N_{22}M_{11}) - 4(N_{11}M_{11} + N_{22}M_{22})]}{h^2} = G_{24}$$

$$G_{16} = \frac{72[N_{12}(2M_{11} - M_{22}) + M_{12}(2N_{11} - N_{22})]}{h^2} = G_{34}$$

$$G_{22} = (2N_{22} - N_{11})^2 + \frac{12(2M_{22} - M_{11})^2}{h^2}$$

$$G_{23} = 6\left[N_{12}(2N_{22} - N_{11}) + \frac{12M_{12}(2M_{22} - M_{11})}{h^2}\right]$$

$$G_{25} = \frac{24(2N_{22} - N_{11})(2M_{22} - M_{11})}{h^2}$$

$$G_{26} = \frac{72[N_{12}(2M_{22} - M_{11}) + M_{12}(2N_{22} - N_{11})]}{h^2} = G_{35}$$

$$G_{33} = 36\left(N_{12}^2 + \frac{12M_{12}^2}{h^2}\right), \quad G_{36} = \frac{864N_{12}M_{12}}{h^2}$$

$$G_{44} = \frac{12[(2N_{11} - N_{22})^2 + 108(2M_{11} - M_{22})^2/5h^2}{h^2}$$

$$G_{45} = \frac{12}{h^2}\left[(2N_{11} - N_{22})(2N_{22} - N_{11}) + \frac{108}{5h^2}(2M_{11} - M_{22})(2M_{22} - M_{11})\right]$$

$$G_{46} = \frac{72[N_{12}(2N_{11} - N_{22}) + 108M_{12}(2M_{11} - M_{22})/5h^2]}{h^2}$$

$$G_{55} = \frac{12[(2N_{22} - N_{11})^2 + 108(2M_{22} - M_{11})^2/5h^2}{h^2}$$

$$G_{56} = \frac{72[N_{12}(2N_{22} - N_{11}) + 108M_{12}(2M_{22} - M_{11})/5h^2]}{h^2}$$

$$G_{66} = \frac{432(N_{12}^2 + 108M_{12}^2/5h^2)}{h^2}$$

It should be noted that in evaluating the integrals leading to the constitutive equations Williams [51] treats E and G as constants. Hence, it is not possible to account for a temperature dependency of E and G without considerable additional complication.

Alternatively, it may be convenient to express the constitutive equation in terms of the symmetric A, B, and D matrices that are defined as follows:

$$A = \begin{bmatrix} 1+HG_{11} & -\nu+HG_{12} & \dfrac{HG_{13}}{2} & \dfrac{HG_{13}}{2} \\ & 1+HG_{22} & \dfrac{HG_{23}}{2} & \dfrac{HG_{23}}{2} \\ & & \dfrac{1+\nu+(HG_{33})/2}{2} & \dfrac{1+\nu+(HG_{33})/2}{2} \\ (sym) & & & \dfrac{1+\nu+(HG_{33})/2}{2} \end{bmatrix}$$

$$B = \begin{bmatrix} G_{14} & G_{15} & \dfrac{G_{16}}{2} & \dfrac{G_{16}}{2} \\ & G_{25} & \dfrac{G_{26}}{2} & \dfrac{G_{26}}{2} \\ & & \dfrac{G_{36}}{4} & \dfrac{G_{36}}{4} \\ (sym) & & & \dfrac{G_{36}}{4} \end{bmatrix}$$

$$D = \begin{bmatrix} \dfrac{12}{h^2}+HG_{44} & -\dfrac{12\nu}{h^2}+HG_{45} & \dfrac{HG_{46}}{2} & \dfrac{HG_{46}}{2} \\ & \dfrac{12}{h^2}+HG_{55} & \dfrac{HG_{56}}{2} & \dfrac{HG_{56}}{2} \\ & & 6\dfrac{1+\nu}{h^2}+\dfrac{HG_{46}}{4} & 6\dfrac{1+\nu}{h^2}+\dfrac{HG_{66}}{4} \\ (sym) & & & 6\dfrac{1+\nu}{h^2}+\dfrac{HG_{66}}{4} \end{bmatrix}$$

where

$$H = \frac{GE}{9h^2}$$

Further, in the neighborhood of a critical point, these matrices can be expanded as

$$A = A^{(0)} + sA^{(1)} + s^2 A^{(2)} + \dots$$
$$B = sB^{(1)} + s^2 B^{(2)} + \dots$$
$$D = D^{(0)} + sD^{(1)} + s^2 D^{(2)} + \dots$$

where

$$A^{(0)} = \begin{bmatrix} 1+g^* & -\nu+g^* & 0 & 0 \\ & 1+g^* & 0 & 0 \\ & & \frac{1+\nu}{2} & \frac{1+\nu}{2} \\ (sym) & & & \frac{1+\nu}{2} \end{bmatrix}$$

$$A^{(1)} = HN_{cr} \begin{bmatrix} 2(2N_{11}^{(1)} - N_{22}^{(1)} + N^{(1)}) & (N_{11}^{(1)} + N_{22}^{(1)} + 2N^{(1)}) & 3N_{12}^{(1)} & 3N_{12}^{(1)} \\ & 2(2N_{22}^{(1)} - N_{11}^{(1)} + N^{(1)}) & 3N_{12}^{(1)} & 3N_{12}^{(1)} \\ & & 0 & 0 \\ (sym) & & 0 & \end{bmatrix}$$

$$B^{(1)} = \frac{12HN_{cr}}{h^2} \begin{bmatrix} 2(2M_{11}^{(1)} - M_{22}^{(1)}) & (M_{11}^{(1)} + M_{22}^{(1)}) & 3M_{12}^{(1)} & 3M_{12}^{(1)} \\ & 2(2M_{22}^{(1)} - M_{11}^{(1)} & 3M_{12}^{(1)} & 3M_{12}^{(1)} \\ & & 0 & 0 \\ (sym) & & & 0 \end{bmatrix}$$

and

$$(D^{(0)}, D^{(1)}) = \frac{12}{h^2} \cdot (A^{(0)}, A^{(1)})$$

As noted in [51] the general form of the matrix $D^{(2)}$ is unnecessarily cumbersome; thus one may anticipate the result that $N_{ij}^{(1)} = N^{(1)} = 0$ and define $D^{(2)}$ as

$$D^{(2)} = \frac{12H}{h^2} \cdot (D_1^{(2)} + D_2^{(2)} + D_3^{(2)})$$

where

$$D_1^{(2)} = \frac{108}{5h^2}\begin{bmatrix} (2M_{11}^{(1)} - M_{22}^{(1)})^2 & (2M_{11}^{(1)} - M_{22}^{(1)})(2M_{22}^{(1)} - M_{11}^{(1)} & 3M_{12}^{(1)}(2M_{11}^{(1)} - M_{22}^{(1)}) & 3M_{12}^{(1)}(2M_{11}^{(1)} - M_{22}^{(1)} \\ & (2M_{22}^{(1)} - M_{11}^{(1)})^2 & 3M_{12}^{(1)}(2M_{22}^{(1)} - M_{11}^{(1)}) & 3M_{12}^{(1)}(2M_{22}^{(1)} - M_{11}^{(1)} \\ & & 9M_{12}^{(1)2} & 9M_{12}^{(1)2} \\ (sym) & & & 9M_{12}^{(1)2} \end{bmatrix}$$

$$D_2^{(2)} = (N)_{cr}\begin{bmatrix} 2(2N_4^{(1)} - N_{22}^{(2)}) & (N_{11}^{(2)} + N_{22}^{(2)}) & 3N_{12}^{(2)} & 3N_{12}^{(2)} \\ & 2(2N_{22}^{(2)} - N_{11}^{(2)}) & 3N_{12}^{(2)} & 3N_{12}^{(2)} \\ & & 0 & 0 \\ (sym) & & & 0 \end{bmatrix}$$

$$D_3^{(2)} = 2(N)_{cr}N^{(2)}\begin{bmatrix} 1 & 1 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 0 & 0 \\ (sym) & & & 0 \end{bmatrix}$$

These results can also be written in index notation:

$$D_{\alpha\beta\gamma\delta}^{(2)} = \frac{12H}{h^2}(D_{\alpha\beta\gamma\delta}^{(2)} + D_{\alpha\beta\gamma\delta}^{(2)} + D_{\alpha\beta\gamma\delta}^{(3)})$$

VIII. INITIAL BUCKLING/POSTBUCKLING FIGURES, GRAPHS, AND TABLES

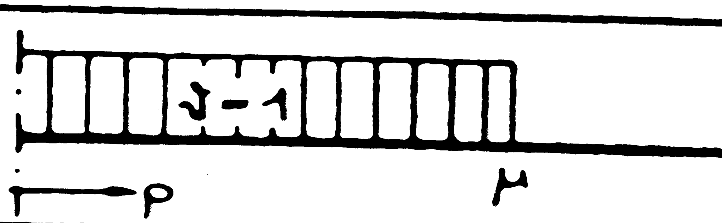
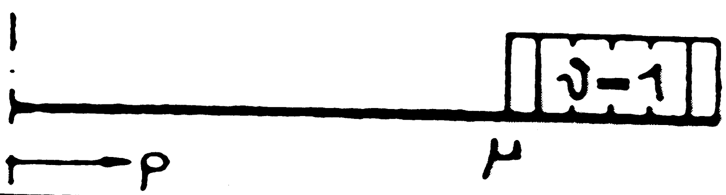
m	μ	λ_R	Temperature Distribution
0	0.63	26.6	
1	0.59	39.2	
2	0.67	20.0	
3	0.69	47.1	

Table 1 Buckling Loads for the Circular Plate [15]

Terms retained	$\dfrac{b^2 E\alpha T_{0_{cr}} t}{\pi^2 D}$
a_{11}	6.35
a_{11}, a_{31}	5.65
a_{11}, a_{31}, a_{13}	5.40
a_{11}, a_{31}, a_{13}, a_{33}	5.39

Table 2 Critical Buckling Temperature for a Rectangular Plate [2]

Material	Lower critical temp.	Upper critical temp.
Elastic	Standard elastic critical temp.	
Maxwell	0	Elastic critical temp.
Kelvin	Elastic critical temp.	∞

Table 3 Upper and Lower Critical Temperatures for Viscoelastic Plates [28]

a/h	$(\bar{N}_T)_{cr}$			
	$\bar{G}$			
	0	0.1	0.2	0.3
0.5	0.2570	0.2553	0.2537	0.2523
1.0	0.4112	0.4046	0.3995	0.3954
1.5	0.6683	0.6448	0.6313	0.6225
2.0	1.028	0.9657	0.9428	0.9320
2.5	1.491	1.365	1.343	1.342
3.0	2.056	1.854	1.857	1.890
3.5	2.724	2.456	2.531	2.632

Table 4 Buckling temperature for simply supported plates ($\nu = 0.3$)

a/b	$-\bar{N}_{cr}$			
	$\bar{G}$			
	0	0.1	0.2	0.3
0.5	0.3672	0.3620	0.3573	0.3530
1.0	0.5875	0.5677	0.5518	0.5385
1.5	0.9546	0.8825	0.8360	0.8020
2.0	1.469	1.265	1.168	1.104
2.5	2.130	1.682	1.522	1.427
3.0	2.937	2.115	1.892	1.767
3.5	3.892	2.562	2.280	2.129

Table 5 Buckling stress for simply supported plates ($\nu = 0.3$)

a/b	$(\bar{N}_T)_{cr}$			
	$\bar{G}$			
	0	0.1	0.2	0.3
0.5	0.3046(1)	0.3017(1)	0.2993(1)	0.2971(1)
1.0	0.7876(1)	0.7529(1)	0.7356(1)	0.7253(1)
1.5	1.743(1)	1.582(1)	1.569(1)	1.579(1)
2.0	3.146(1)	2.858(1)	2.999(1)	3.166(1)
2.5	4.831(2)	4.696(2)	5.314(2)	5.931(2)
3.0	6.973(2)	7.744(2)	9.583(2)	11.37(2)
3.5	9.491(2)	12.68(3)	17.17(3)	21.54(3)

Table 6 Buckling temperature for mixed boundary condition plates ($\nu = 0.3$)

a/h	$-\bar{N}_{cr}$			
	$\bar{G}$			
	0	0.1	0.2	0.3
0.5	0.4351(1)	0.4266(1)	0.4192(1)	0.4126(1)
1.0	1.125(1)	1.016(1)	0.9532(1)	0.9091(1)
1.5	2.490(1)	1.883(1)	1.693(1)	1.584(1)
2.0	4.494(1)	2.819(1)	2.507(1)	2.344(1)
2.5	6.902(2)	3.746(2)	3.351(2)	3.156(2)
3.0	9.961(2)	4.806(2)	4.359(2)	4.151(2)
3.5	13.56(3)	5.992(3)	5.524(3)	5.320(3)

Table 7 Buckling stress for mixed boundary condition plates ($\nu = 0.3$)

$\bar{G}$	$-\bar{N}_{cr}$	$(\bar{N}_T)_{cr}$
0	0.2937	0.2056
0.1	0.2910	0.2047
0.2	0.2885	0.2039
0.3	0.2861	0.2031

Table 8 Buckling stress and temperature for the simply supported strip ($\lambda = \pi/2$)

$\bar{G}$	$-\bar{N}_{cr}$	$(\bar{N}_T)_{cr}$
0	1.175	0.8225
0.1	1.054	0.7841
0.2	0.9860	0.7658
0.3	0.9390	0.7553

Table 9 Buckling stress and temperature for the clamped strip ($\lambda = \pi$)

$\tilde{G}$	$-\tilde{N}_{cr}$	$N^{(2)}/N_{cr}$	$-N_{11}^{(2)}/N_{cr}$	$-\dfrac{N_{11}^{(2)}+N^{(2)}}{N_{cr}}$
0.1	1.054	3.004	3.786	0.7822
0.2	0.9860	3.069	4.279	1.2103
0.3	0.9390	3.120	4.653	1.5327

Table 10 Curvature parameters for the clamped strip ($\lambda = \pi, \nu = 0.3$)

$\bar{G}$	$-\bar{N}_{cr}$	$N^{(2)}/N_{cr}$	$-N_{11}^{(2)}/N_{cr}$	$-\dfrac{N_{11}^{(2)}+N^{(2)}}{N_{cr}}$
0.1	0.2910	2.991	3.067	0.0762
0.2	0.2885	2.984	3.131	0.1462
0.3	0.2861	2.980	3.191	0.2107

Table 11 Curvature parameters for the simply supported strip ($\lambda = \pi/2, \nu = 0.3$)

$\bar{G}$	$\eta(ss)$	(a) $\frac{N_{cr}(\bar{G})}{N_{cr}(0)}$	$\eta(c)$	(b) $\frac{N_{cr}(\bar{G})}{N_{cr}(0)}$
0.1	0.6383	0.6583	0.8982	0.8970
0.2	0.5342	0.5858	0.8366	0.8391
0.3	0.4701	0.5470	0.7928	0.7991

Table 12 Comparison with mechanically loaded plates

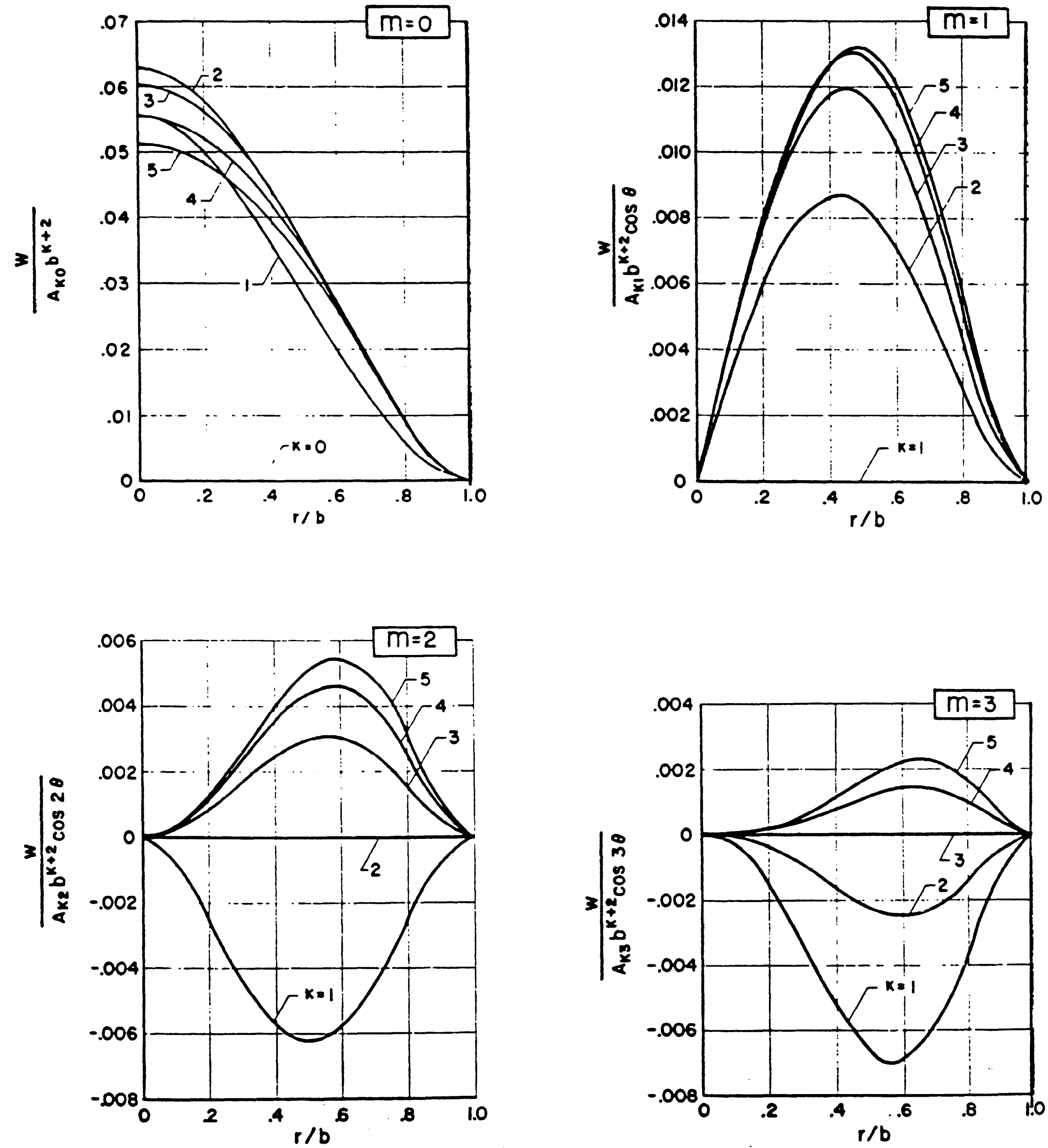


Fig. 1 Nondimensional Deflections as a Function of Distance from the Center of the Plate [11]

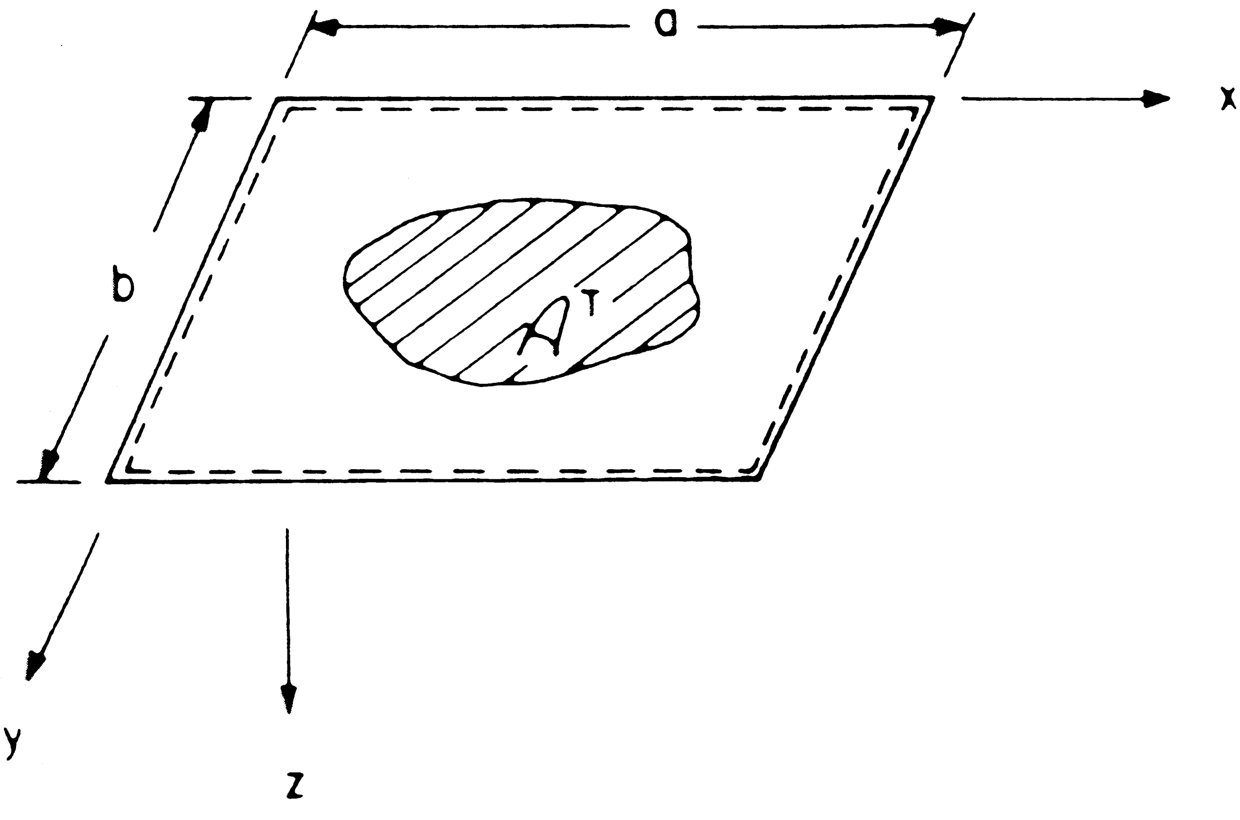


Fig. 2 **Plate Containing a Heated Region $\mathcal{A}^T$ [3]**

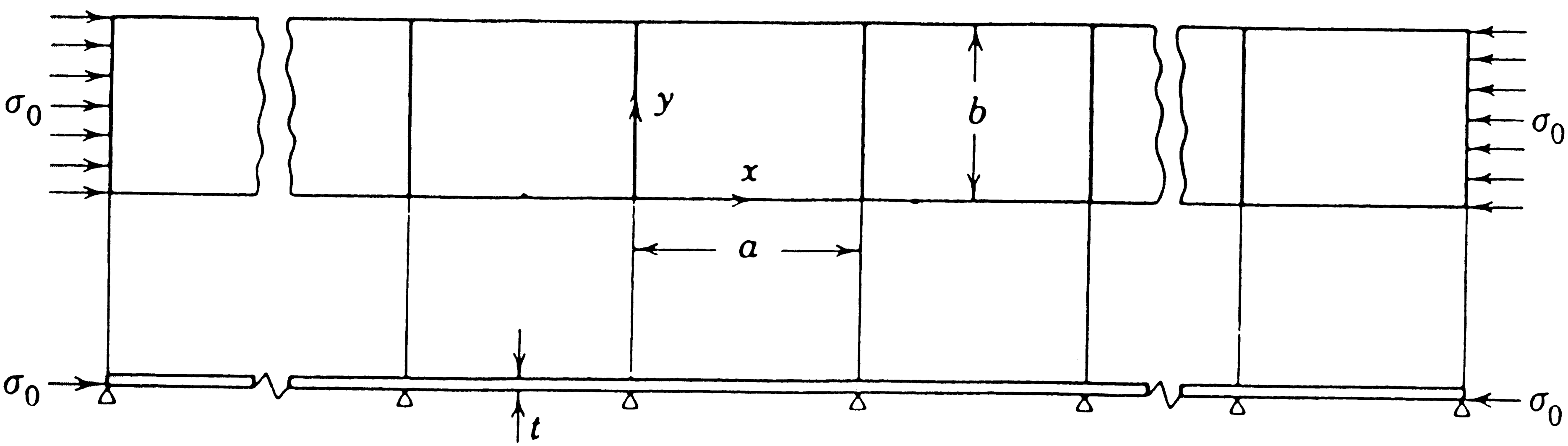


Fig. 3 Buckling of a Heated Plate Strip [7]

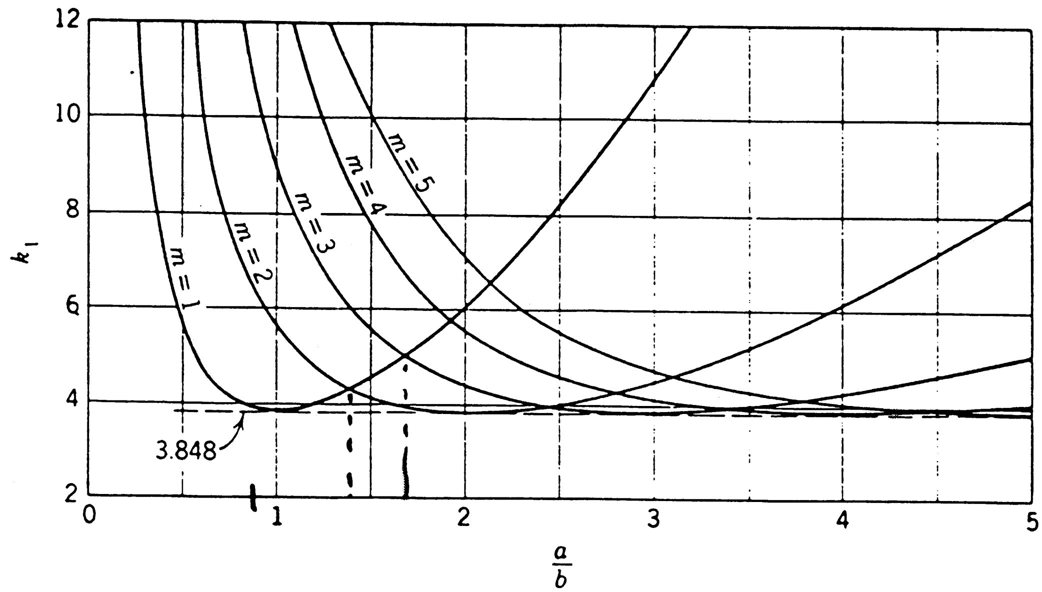


Fig. 4 **Critical Buckling Temperature Coefficients** $\frac{\alpha E}{2} T_{cr} = \frac{k\pi^2 E}{12(1-\nu^2)} \left(\frac{h}{b}\right)^2$ [7]

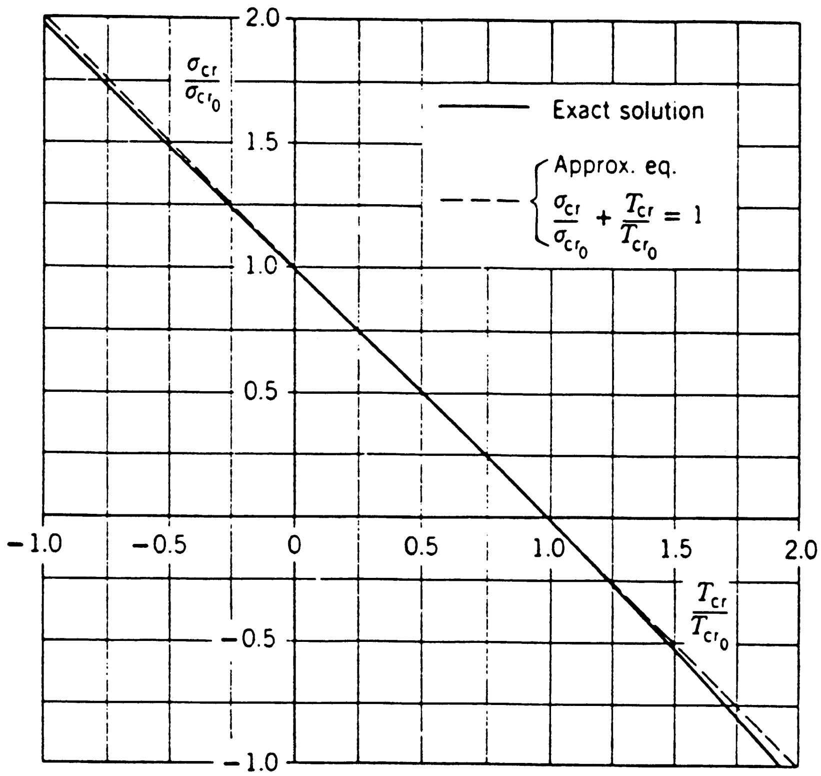


Fig. 5 **Interaction of Temperature Level and Applied Stress in Plate Buckling [7]**

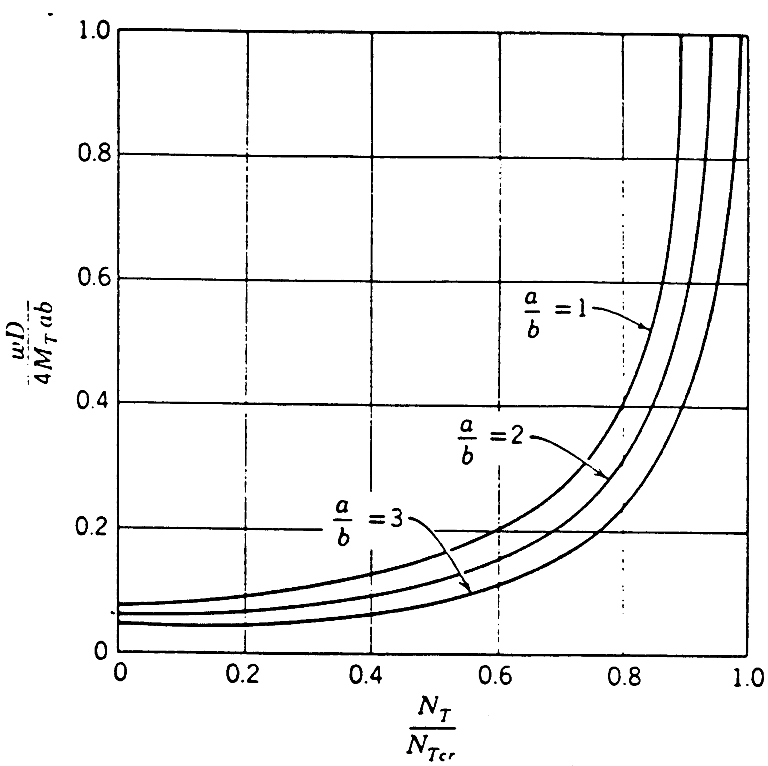


Fig. 6 **Deflection w at the Center of the Rectangular Plate; Poisson Ratio $\nu = 1/4$ [7]**

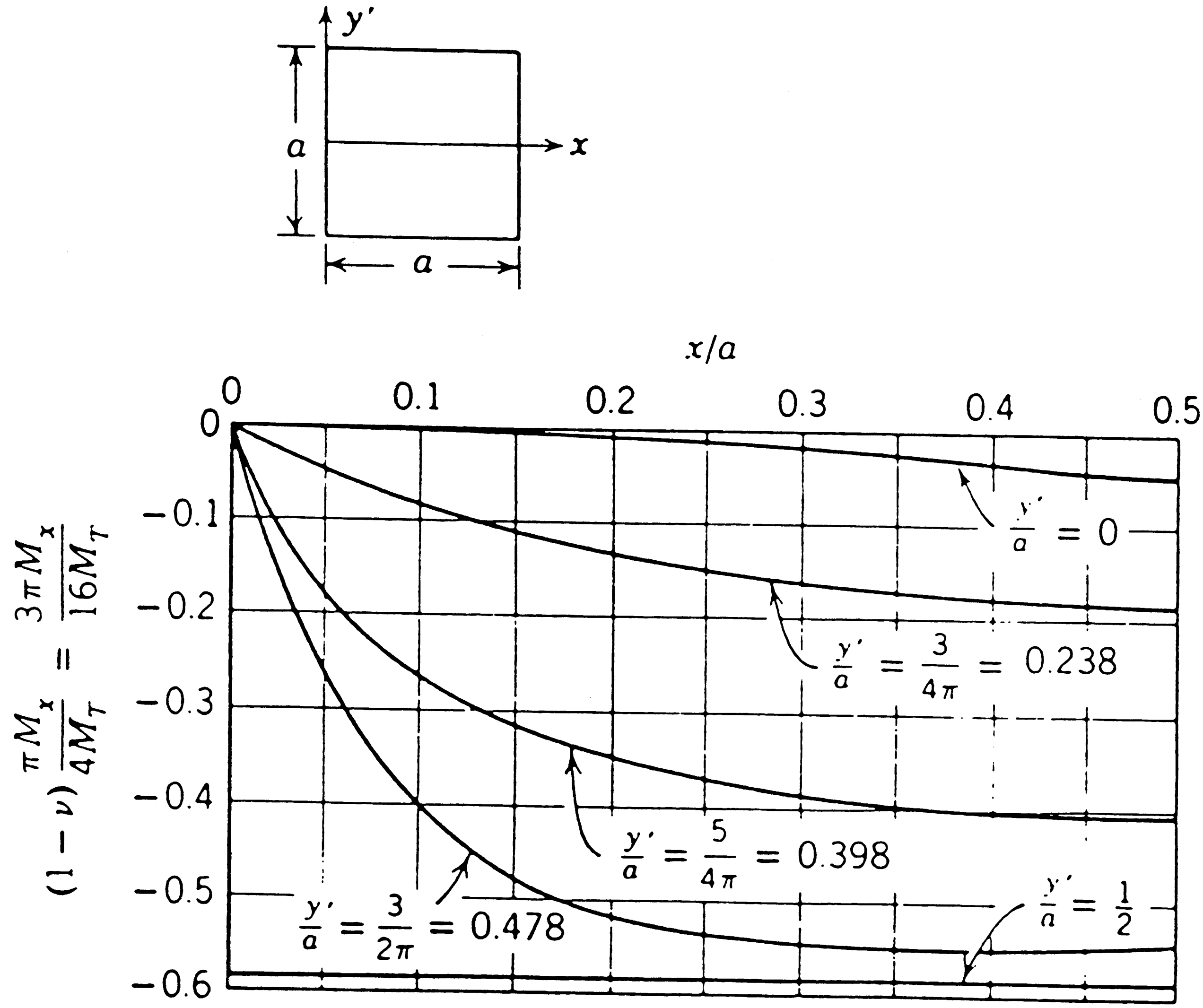


Fig. 7 **Distribution of the Bending Moment M_x in a Square Plate; $N_T/(N_T)_{cr} = 0.25$, $\nu = \frac{1}{4}$ [7]**

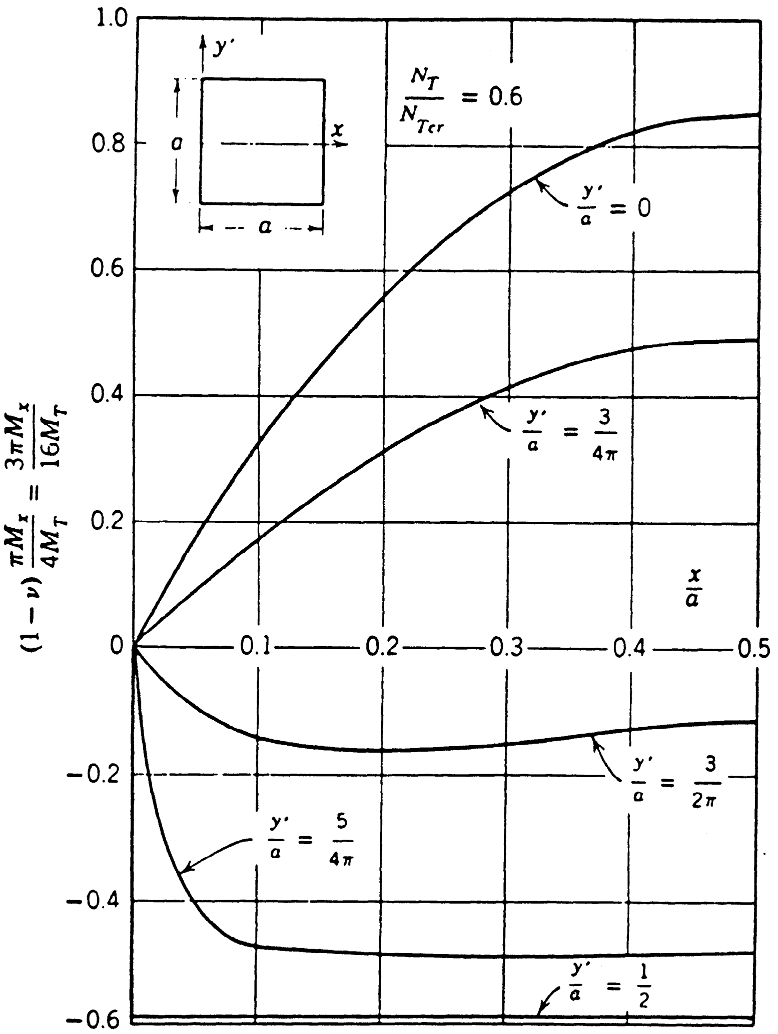


Fig. 8 **Distribution of the Bending Moment M_x in a Square Plate; $N_T/(N_T)_{cr} = 0.6$, $\nu = \frac{1}{4}$ [7]**

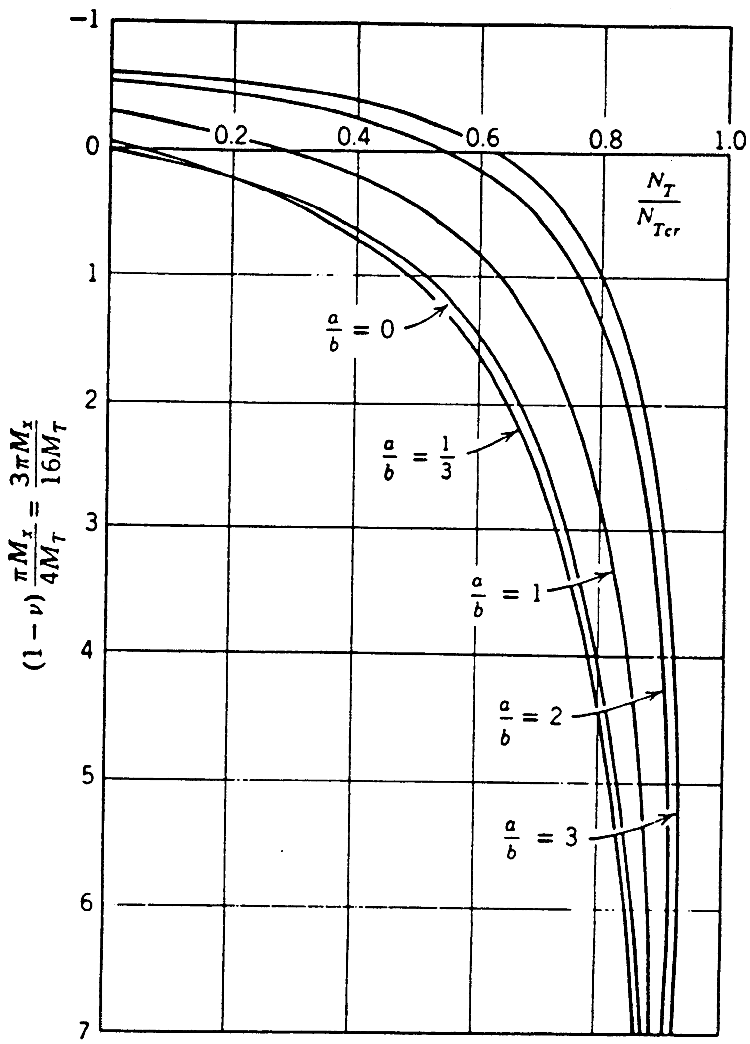


Fig. 9 Bending Moment M_x at the Center of the Rectangular Plate [7]

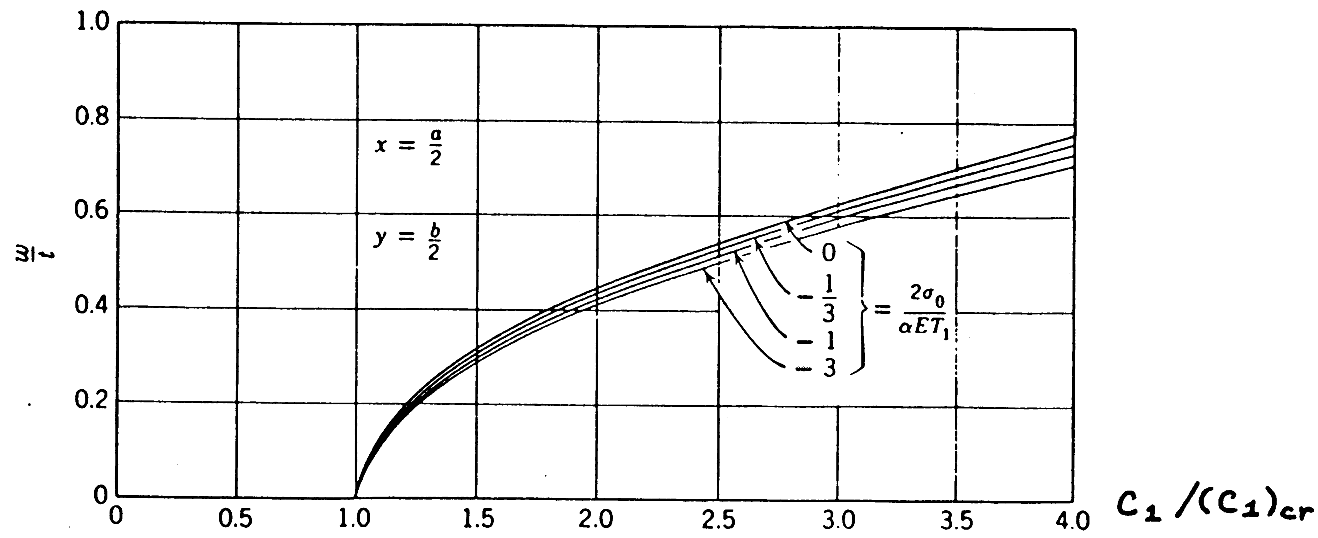


Fig. 10 **Deflection at the Center of the Buckled Plate [7]**

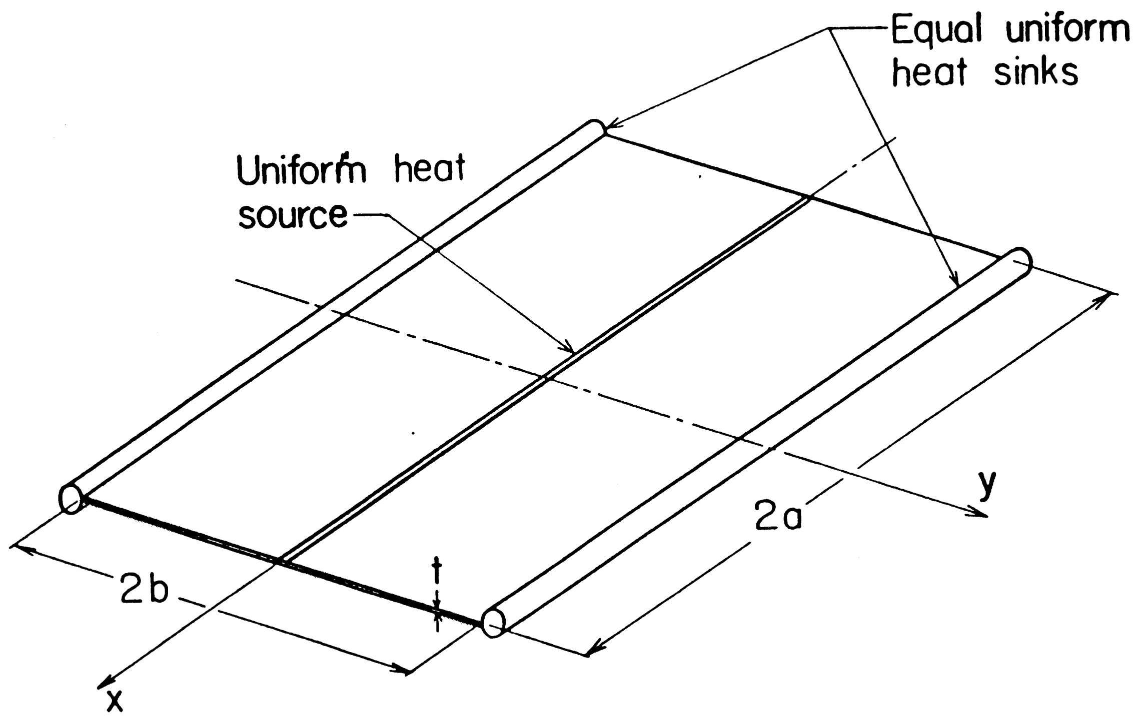


Fig. 11a The Thermal Buckling Problem of Ref. [2]

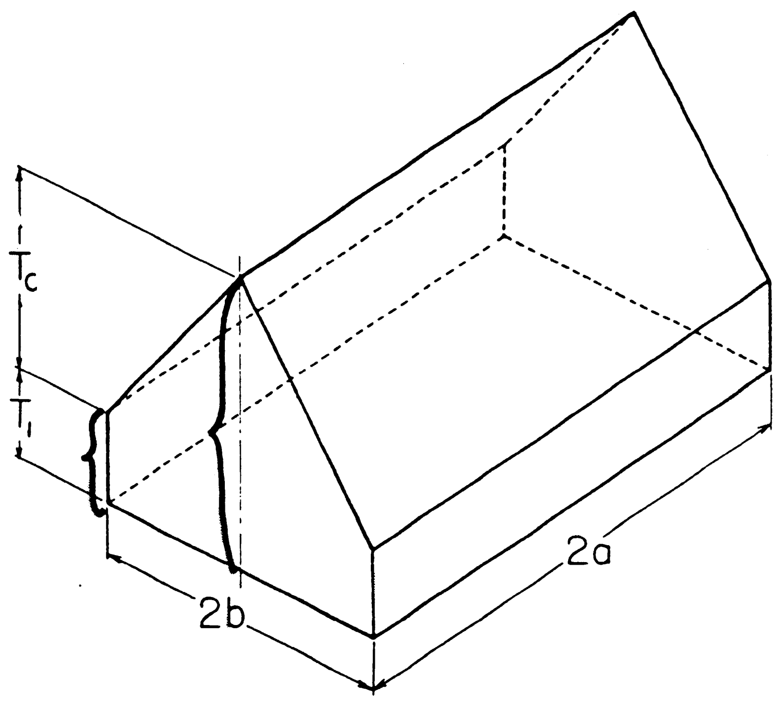


Fig. 11b The Tentlike Temperature Distribution [2]

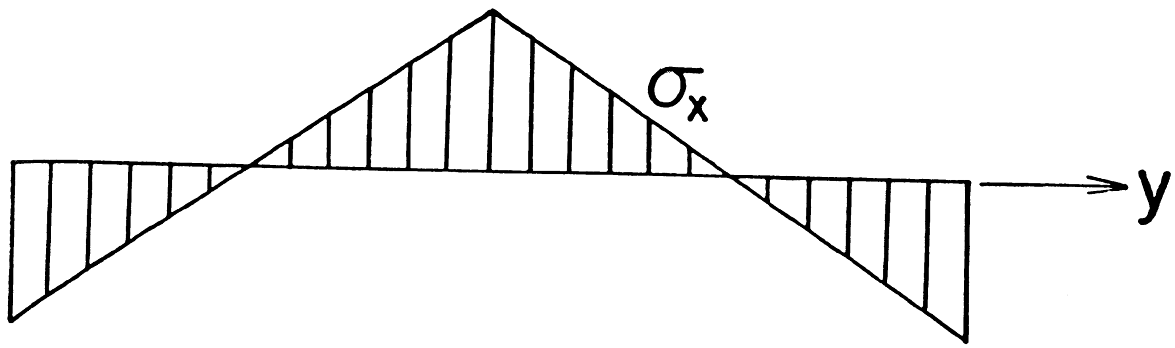


Fig. 12 **Variation of the Normal Stress σ_x [2]**

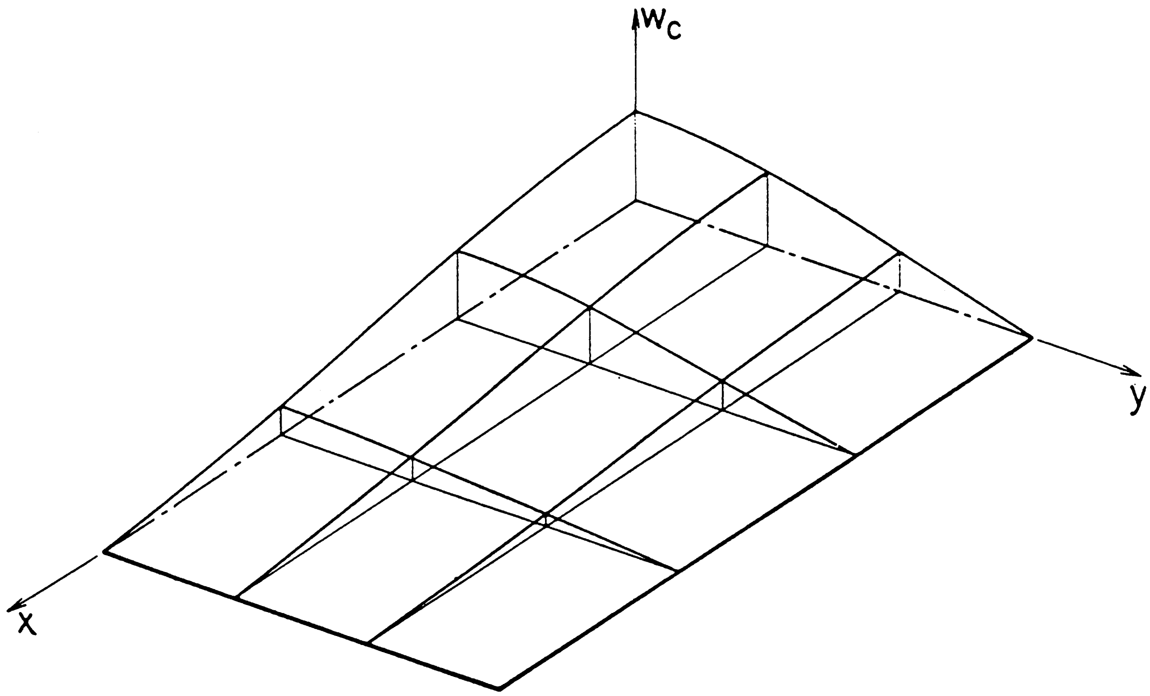


Fig. 13 Small-Deflection Buckle Pattern in One Quadrant of a Plate of Aspect Ratio 1.57 [2]

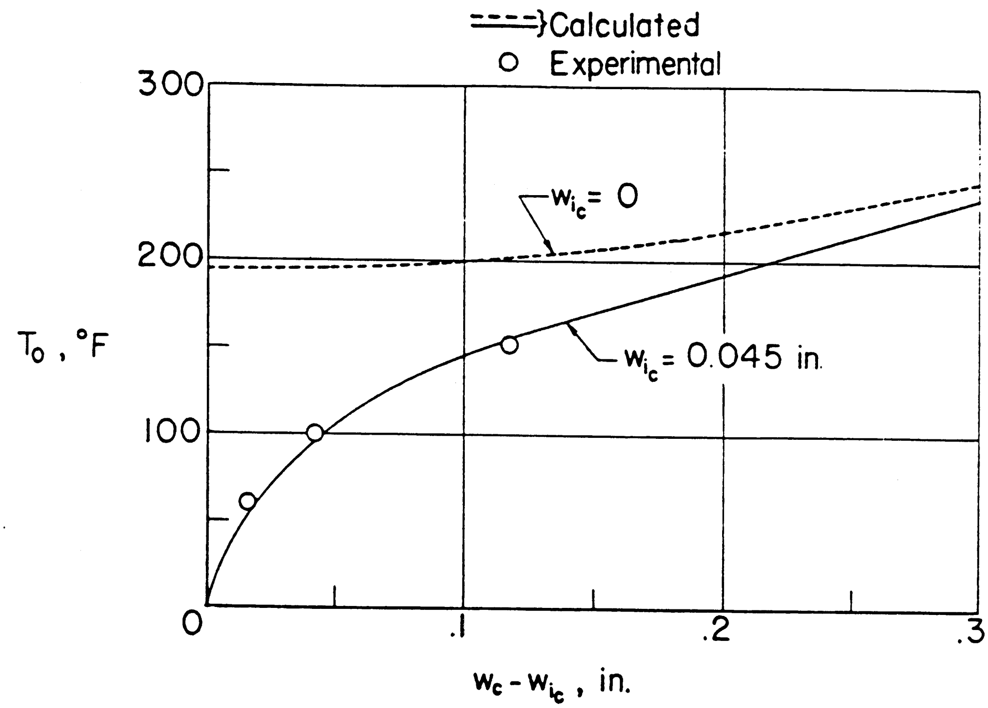


Fig. 14 Comparison of Calculated and Experimental Deflections at the Plate Center [2]

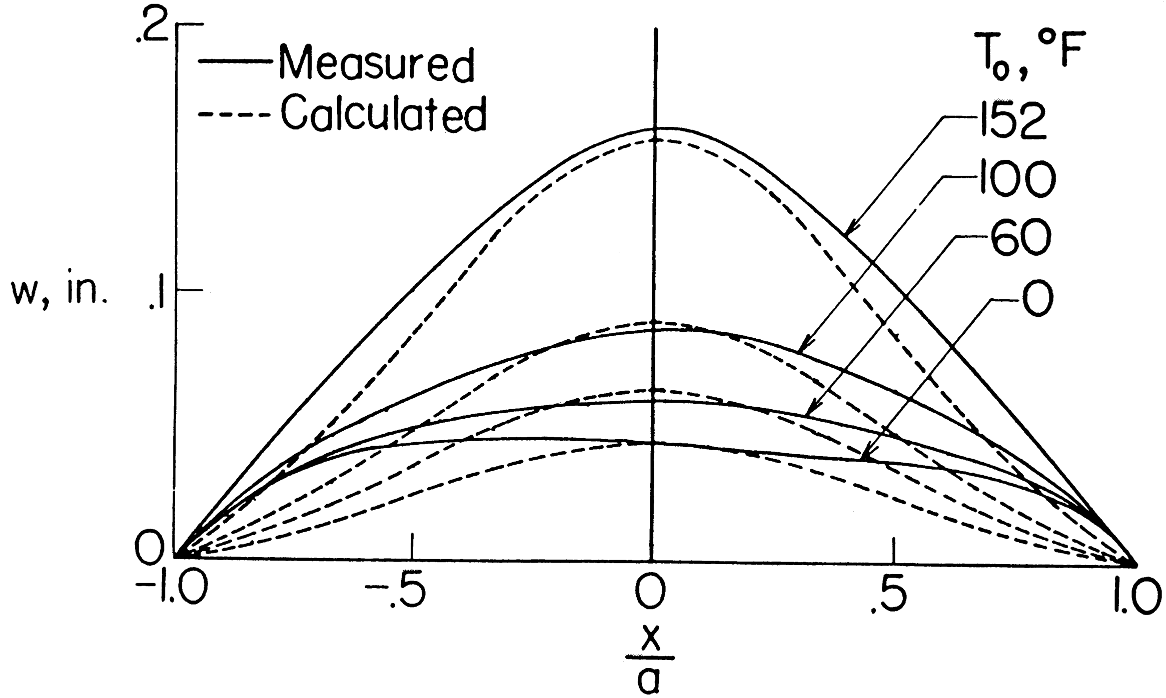


Fig. 15a Longitudinal Center Line Deflection [2]

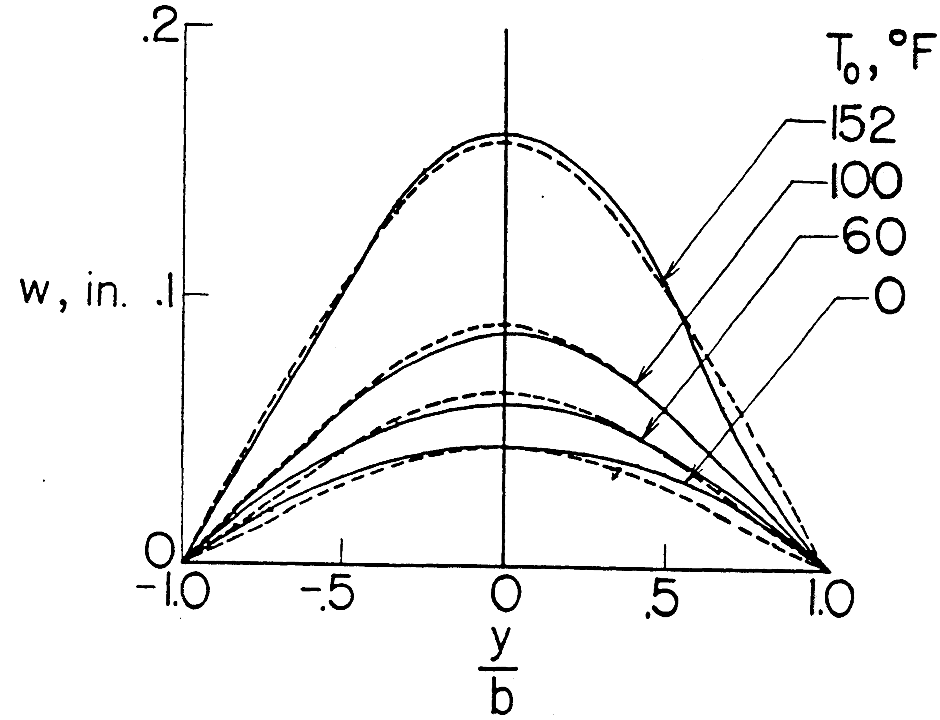


Fig. 15b Growth of Deflections with Temperature [2]

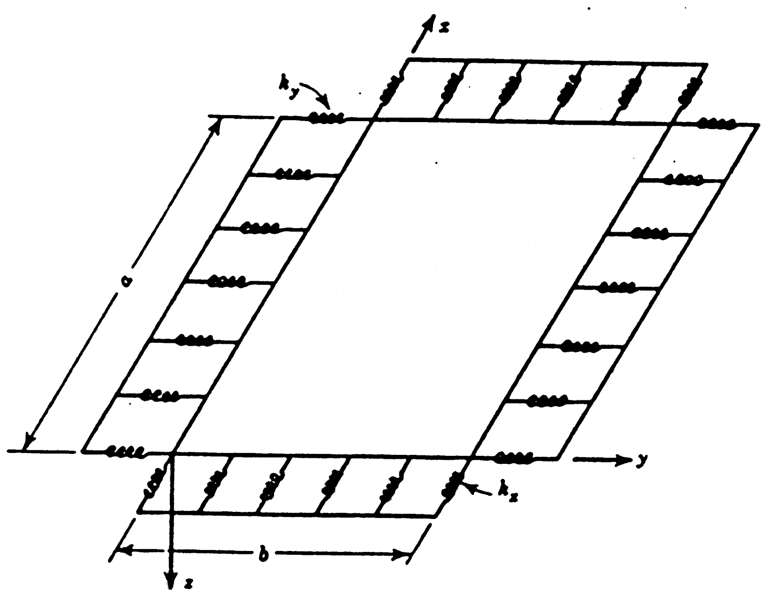


Fig. 16 Rectangular Plate with Edge Restraint in the Plane of the Plate [20]

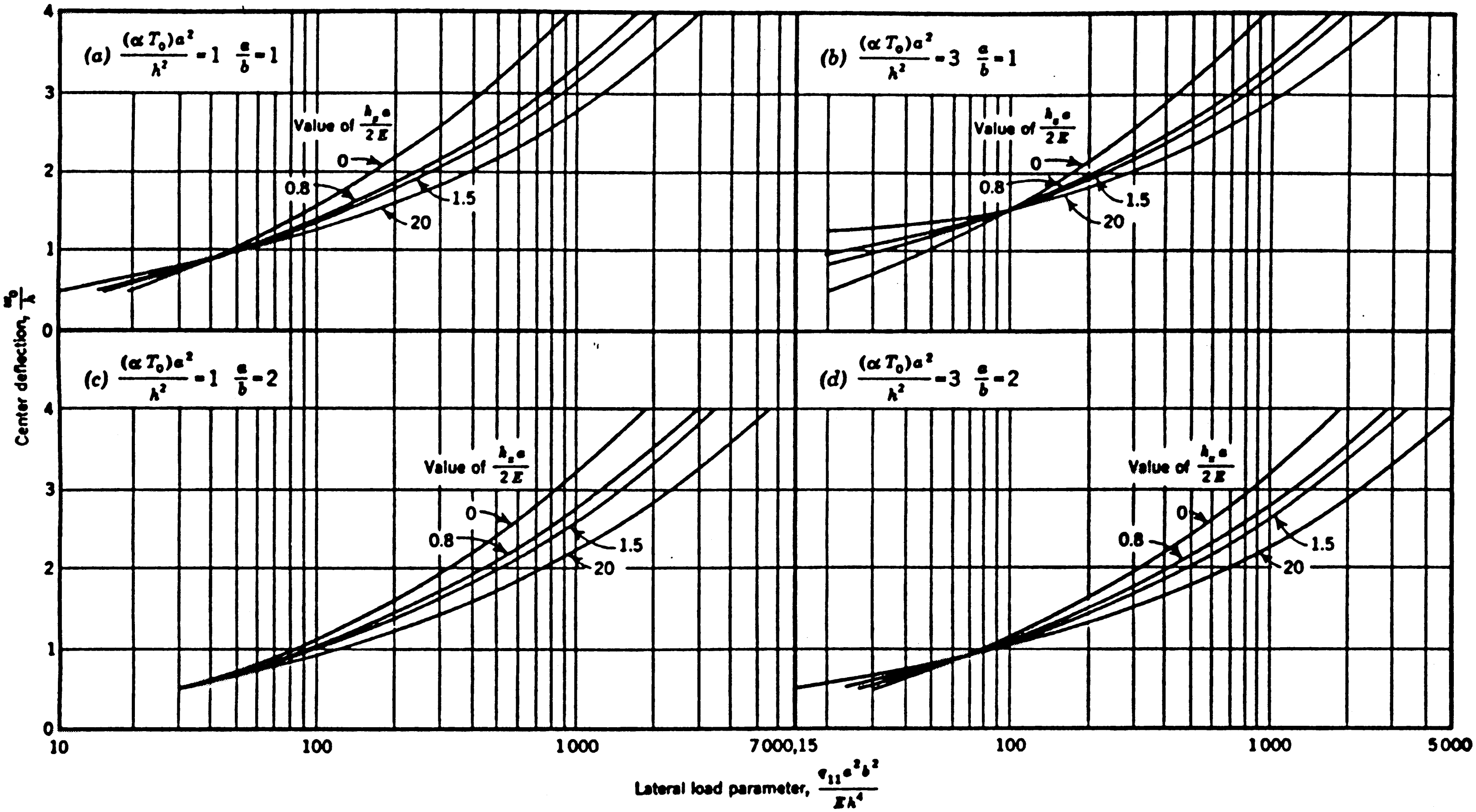


Fig. 17 Center Stress Load Sinusoidal Load Distribution [20]

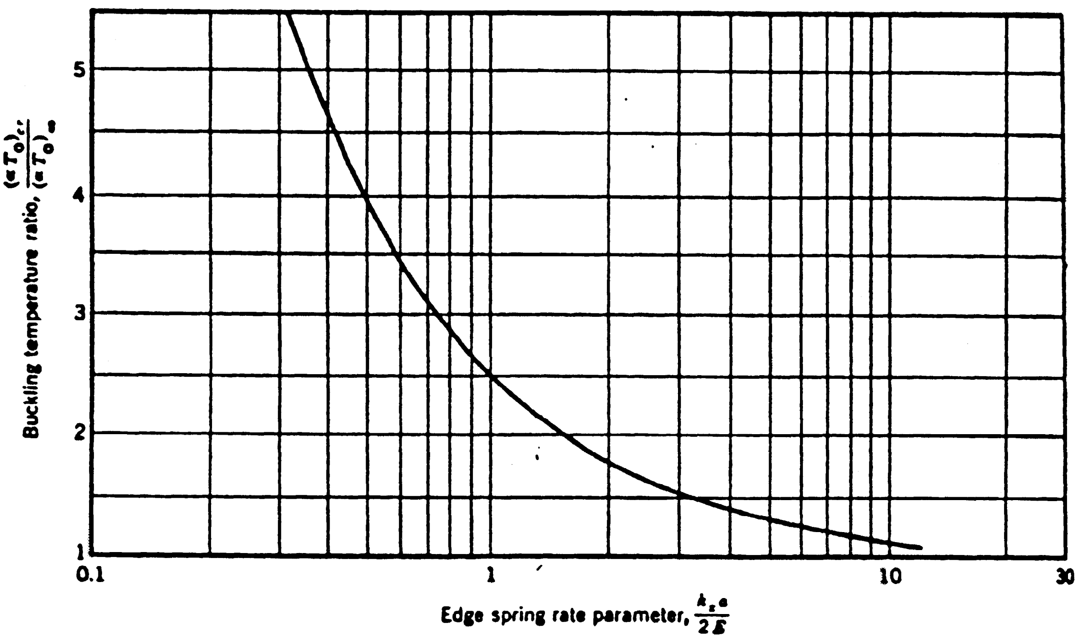


Fig. 18 Effect of Edge Spring Rate on Buckling Temperature Ratio [20]

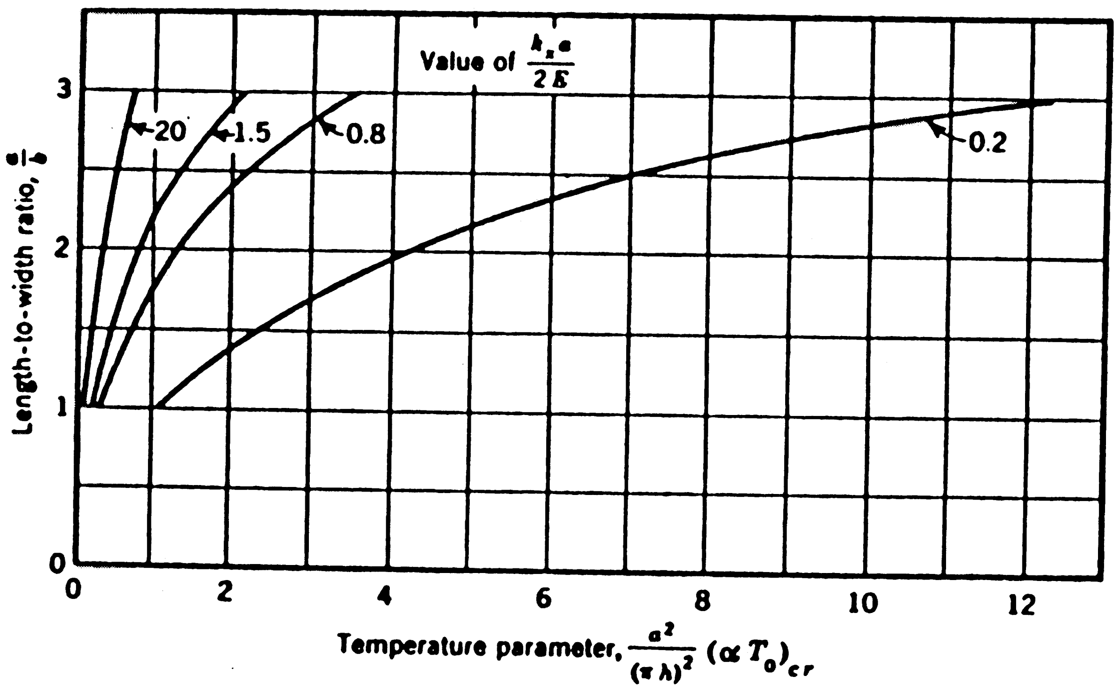


Fig. 19 Effect of Edge Fixity on Buckling Temperature [20]

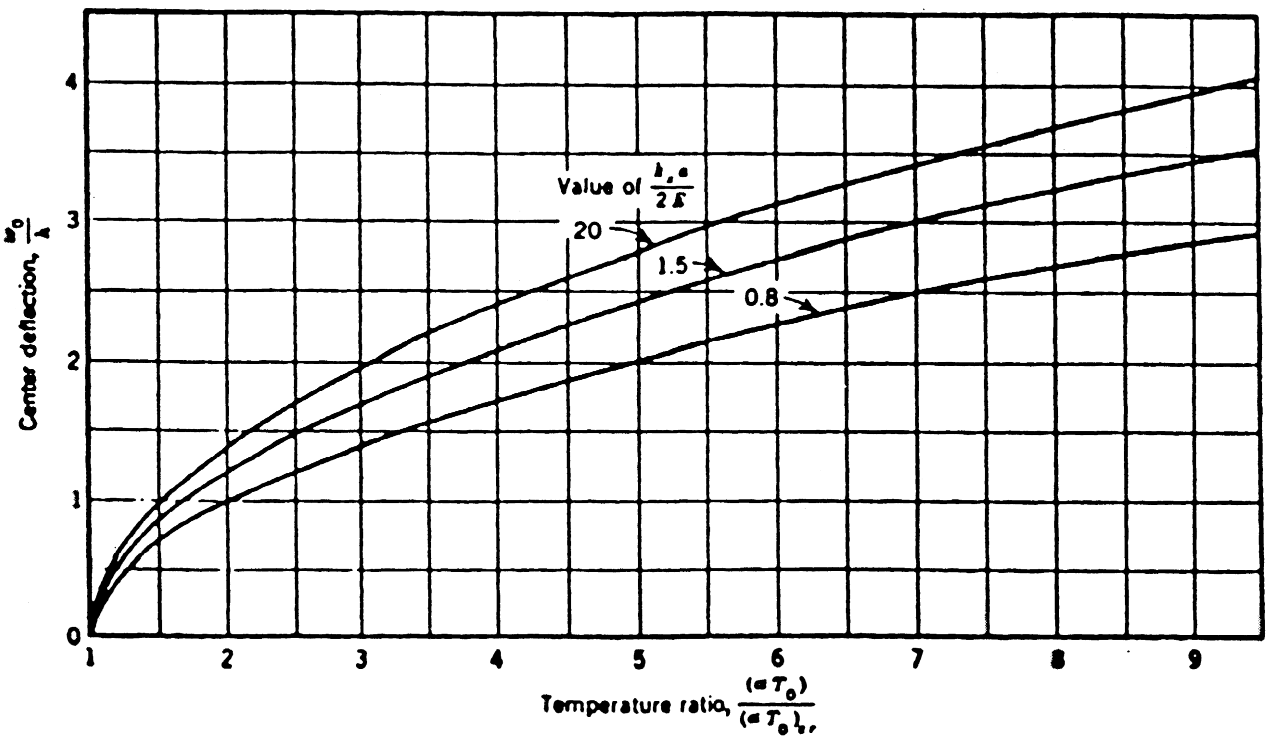


Fig. 20 Center Deflection vs. Temperature Ratio, $\frac{a}{b} = 1$ [20]

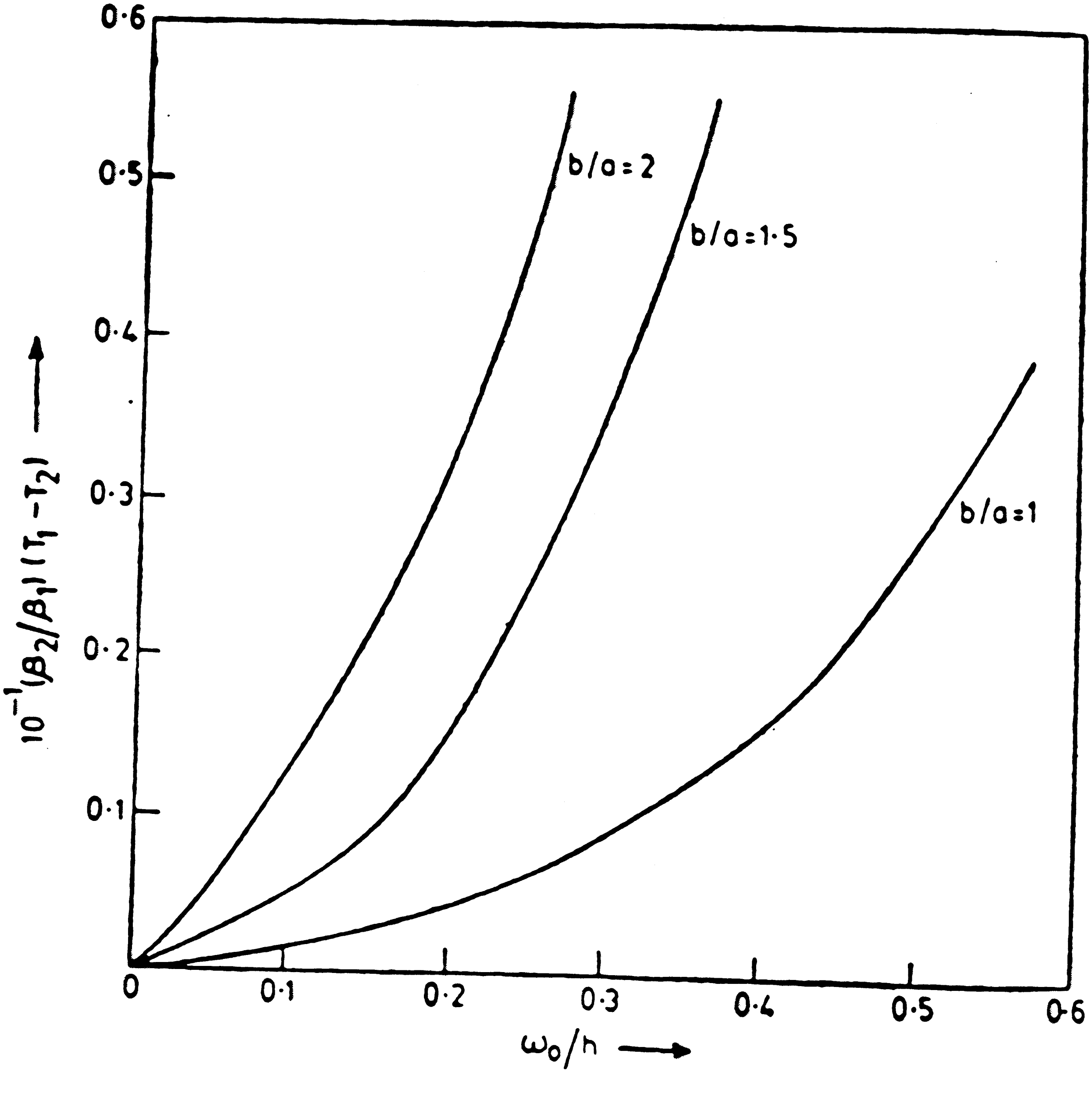


Fig. 21 Variation of Non-Dimensional Deflections w_o/h for Different Values of the Temperature Parameter [22]

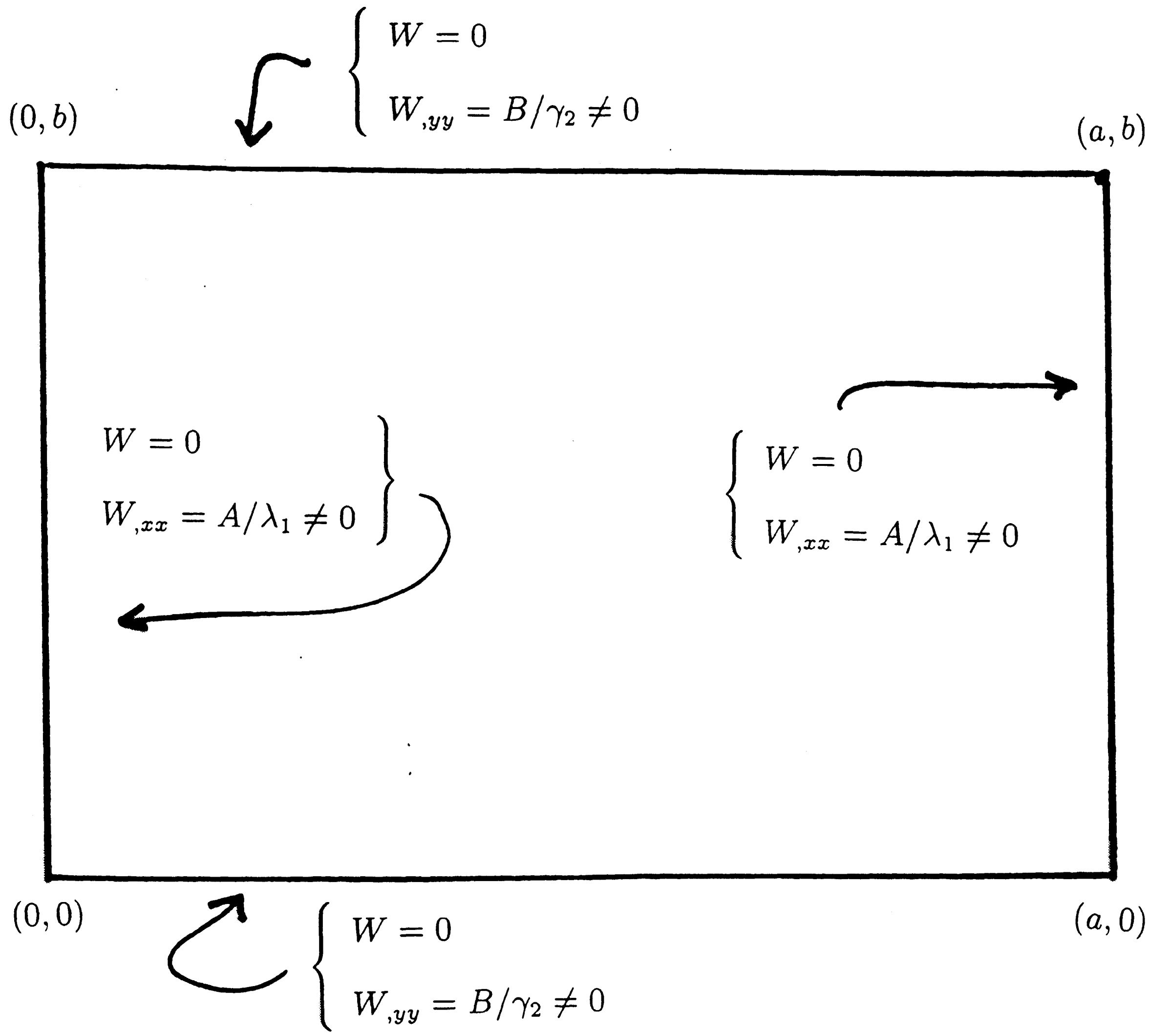


Fig. 22 **Simply Supported Rectangular Plate with a Non-zero Thermal Moment**

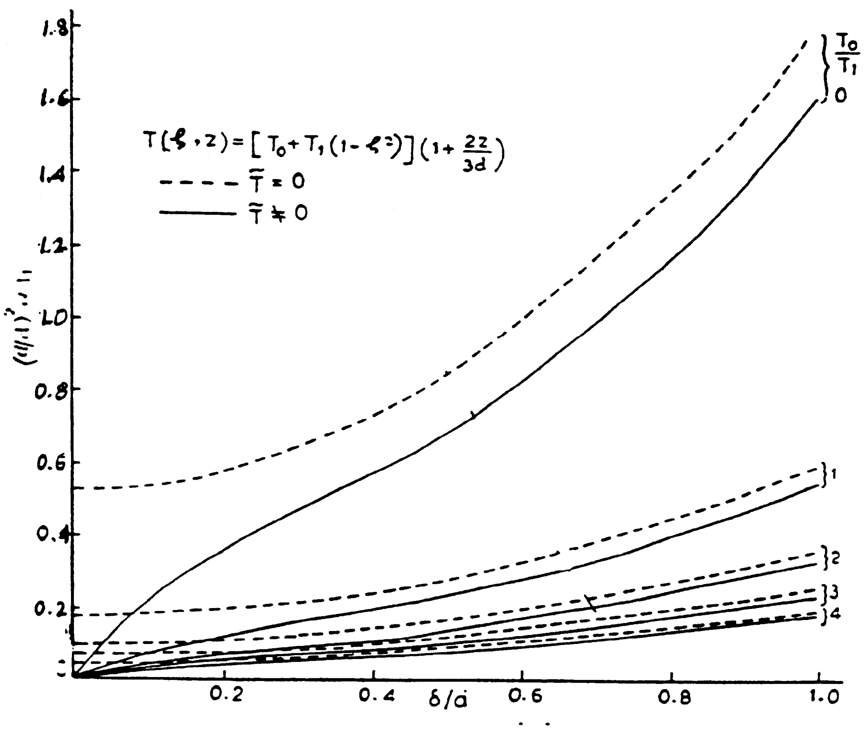


Fig. 23 Relation Between the Temperature Rise and the Deflection (Simply Supported Edge) [32]

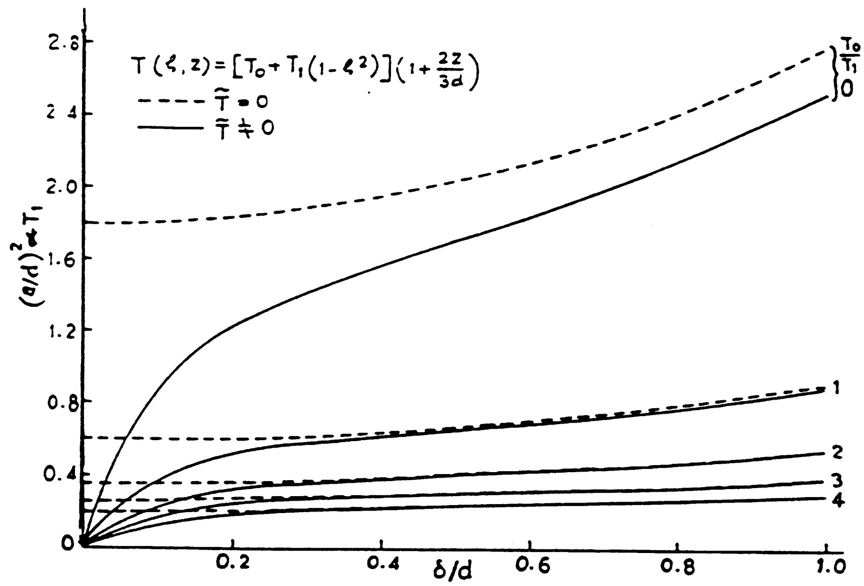


Fig. 24 Relation Between the Temperature Rise and the Deflection (Clamped Edge) [32]

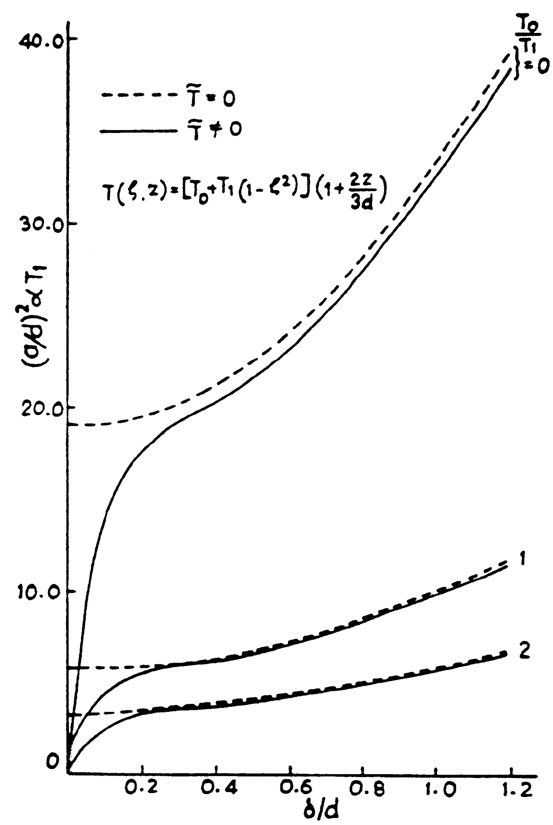


Fig. 25 **Relation Between the Temperature Rise and the Deflection; $\frac{b}{a} = 0.4$ with Edges Simply Supported [32]**

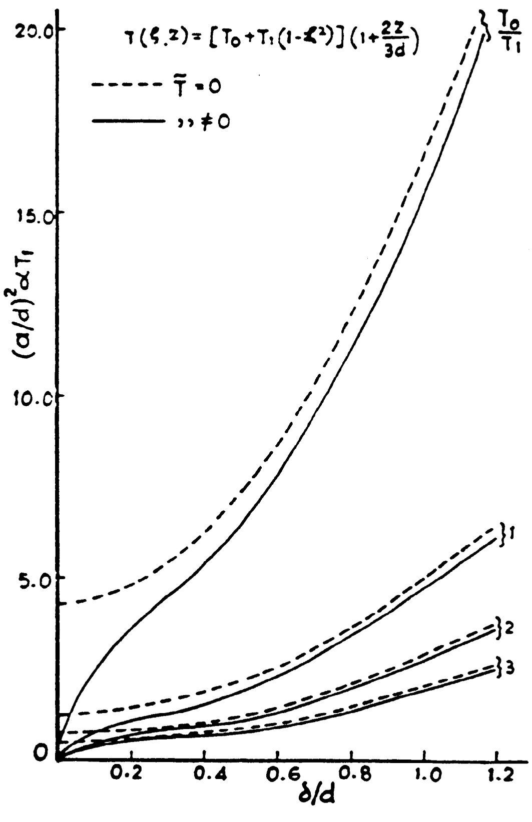


Fig. 26 Relation Between the Temperature Rise and the Deflection; $\frac{b}{a} = 0.4$ with Both Edges Clamped [32]

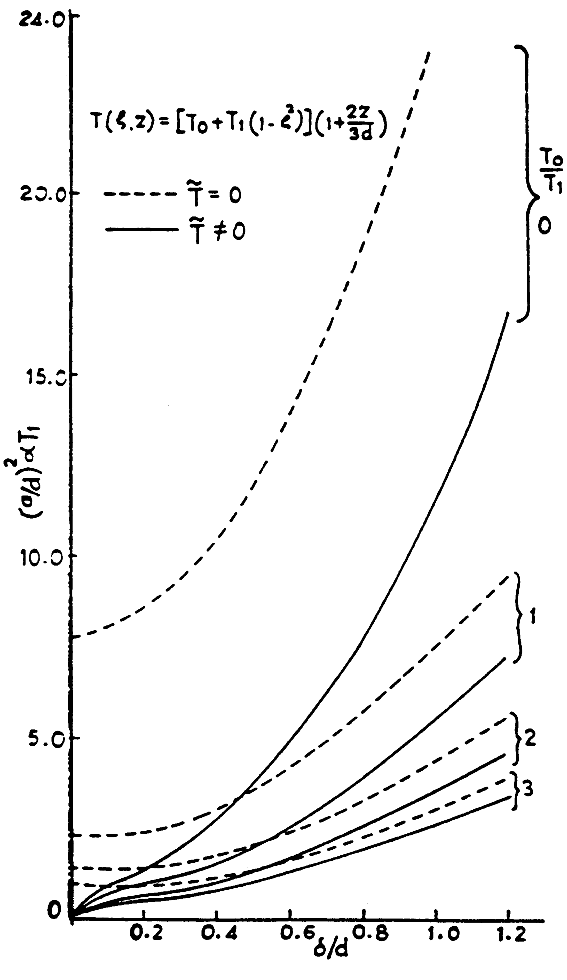


Fig. 27 Relation Between the Temperature Rise and the Deflection; $\frac{b}{a} = 0.4$ with the Inner Edge Clamped and the Outer Edge Simply Supported [32]

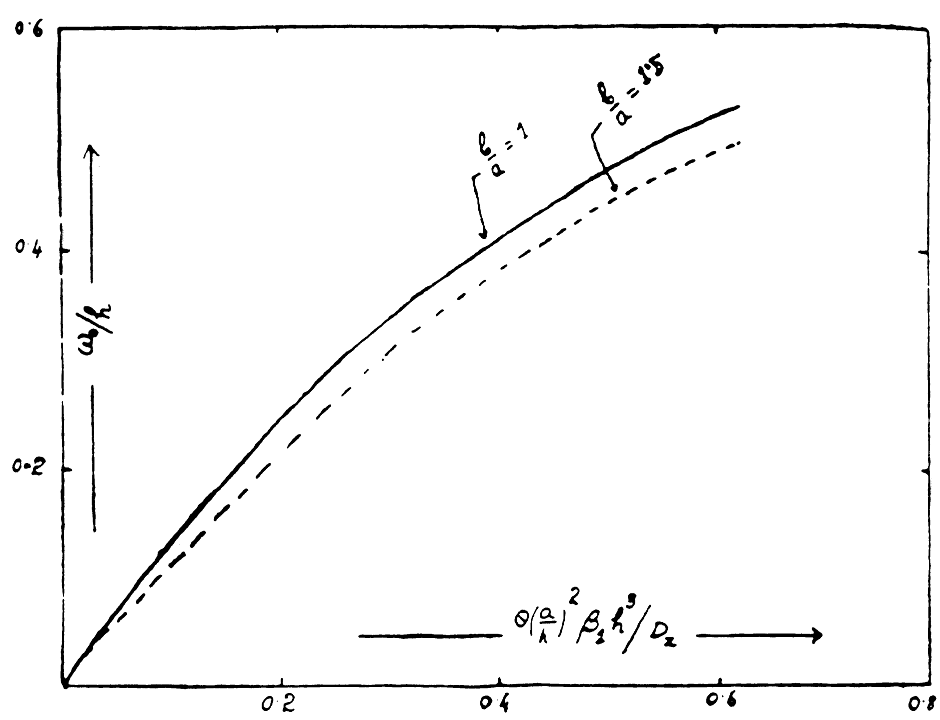


Fig. 28 Variation of Central Deflections with the Temperature Parameter [33]

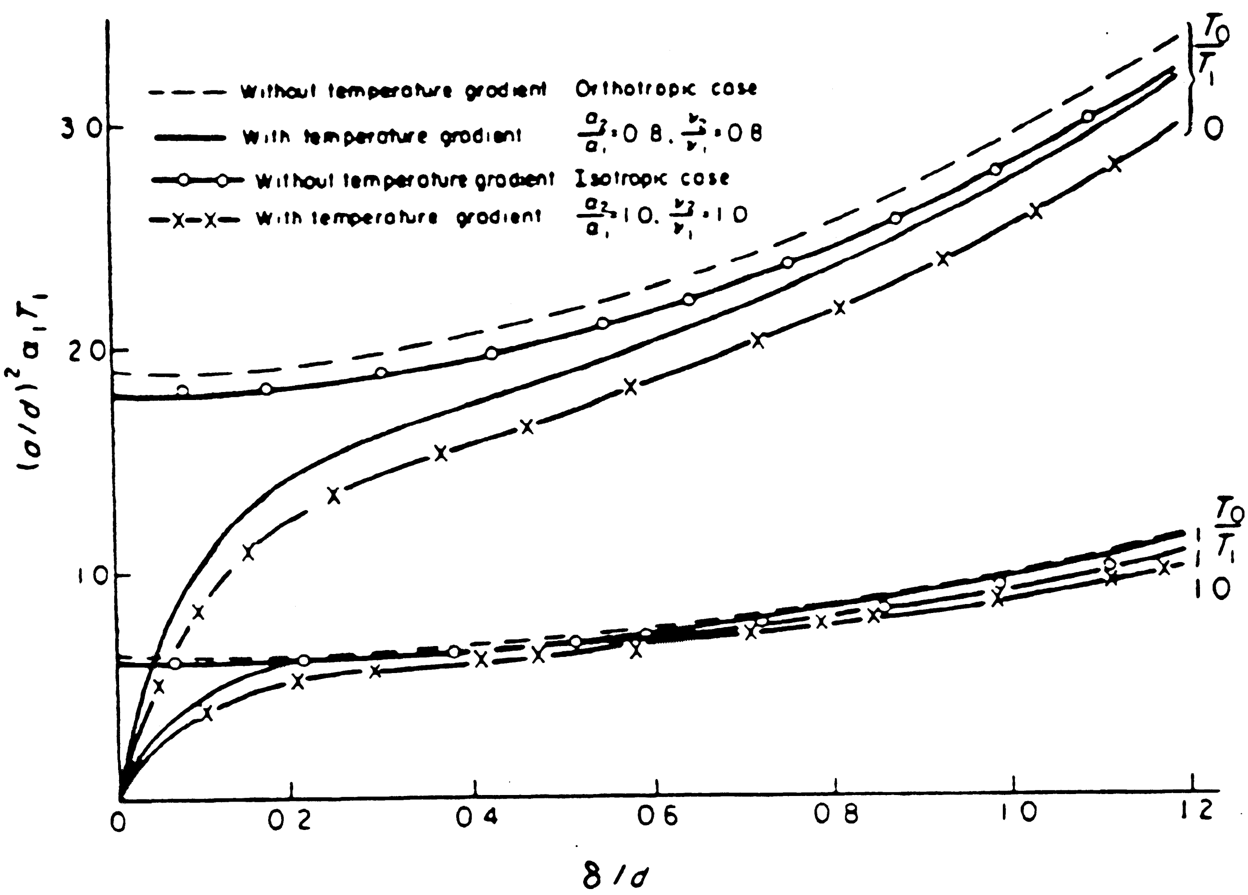


Fig. 29 Variation of Central Deflections with and without a Temperature Gradient [36]

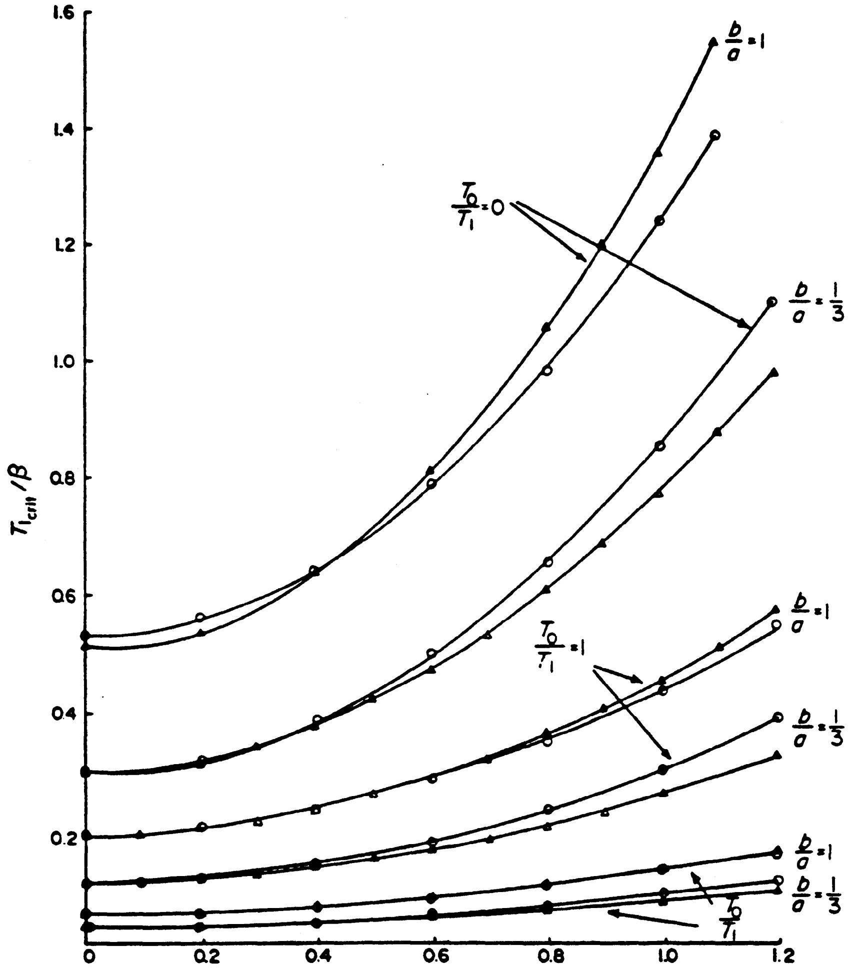


Fig. 30 Comparison of Results for Central Deflection vs. Temperature Rise: $b/a = 1.0,\ 1/3,\ \beta = 1/(\alpha(1-\nu^2))(h/b)^2$ [28]

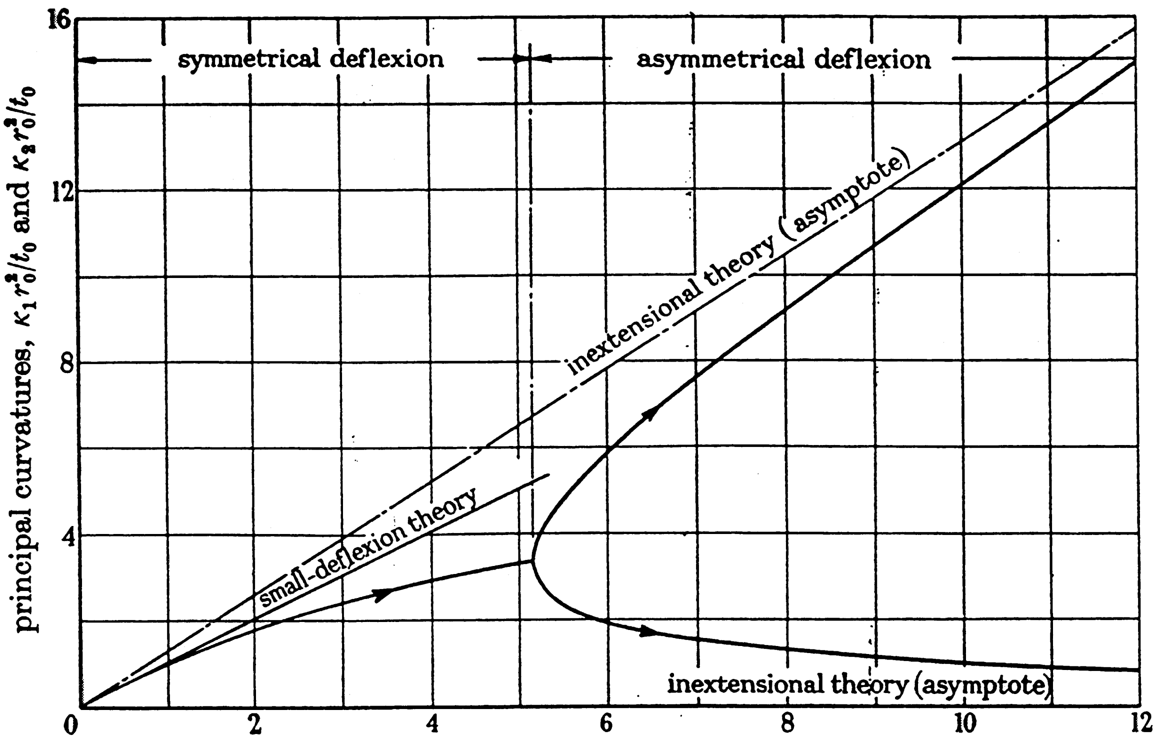


Fig. 31 **Variation of Principal Curvatures with the Temperature Gradient through the Thickness ($k_0 = 0$). [42]**

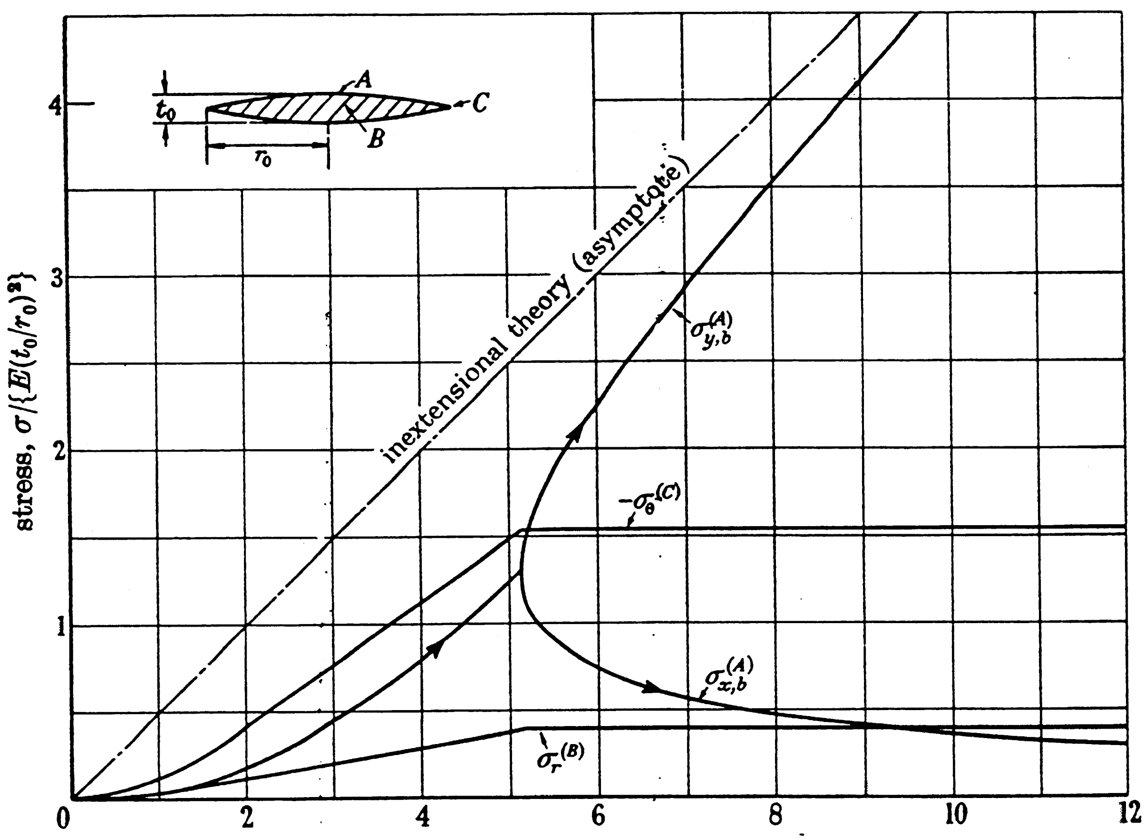


Fig. 32 Variation of Middle-Surface and Bending Stresses with the Temperature Gradient through the Thickness ($\hat{k}_0 = 0$). [42]

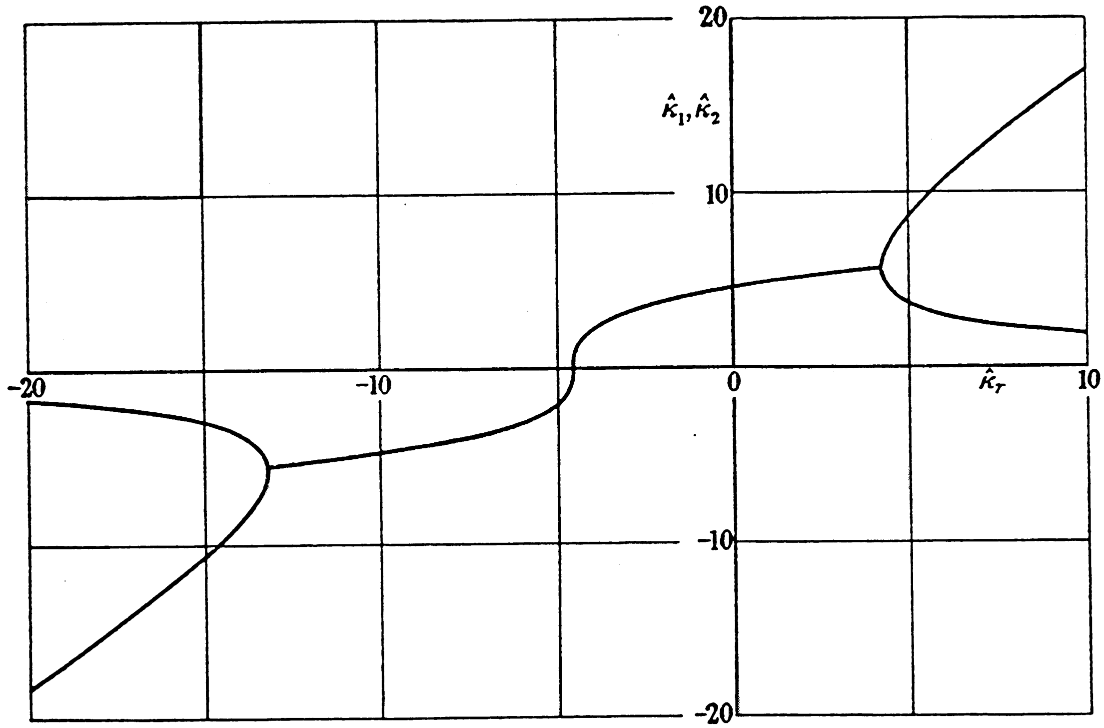


Fig. 33 Variation of Principal Curvatures with the Temperature Gradient through the Thickness ($\hat{k}_0 = \lambda^{1/2}$). [42]

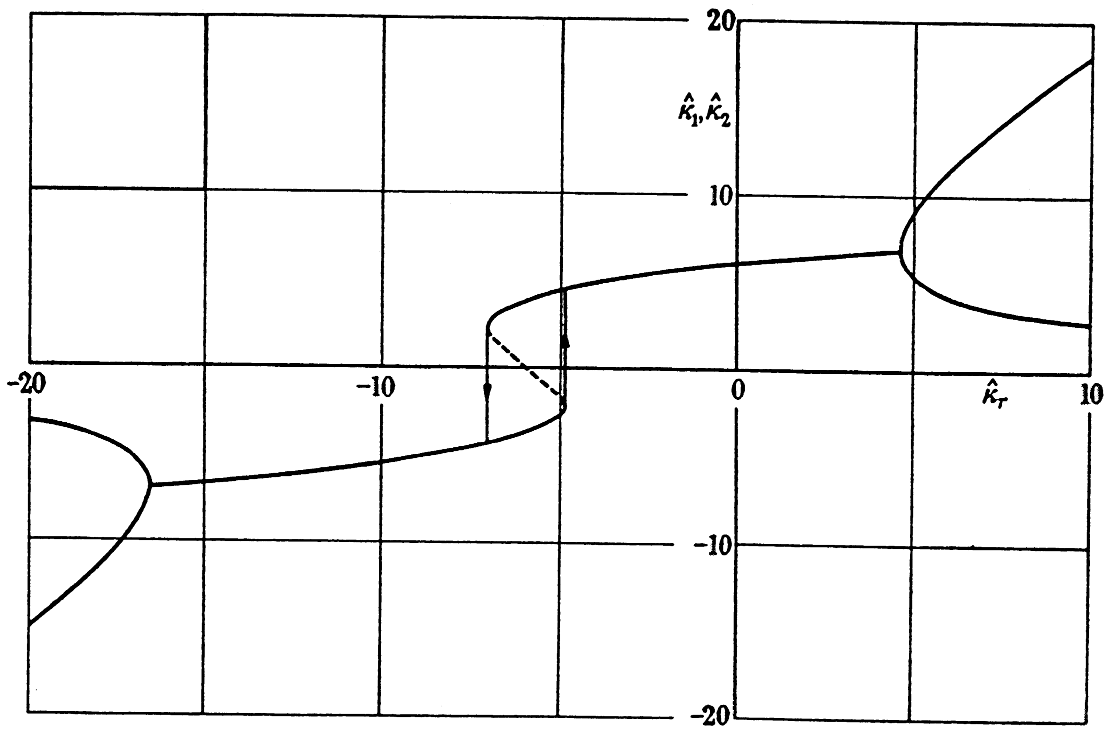


Fig. 34 Variation of Principal Curvatures with the Temperature Gradient through the Thickness ($\hat{k}_0 = 6$). [42]

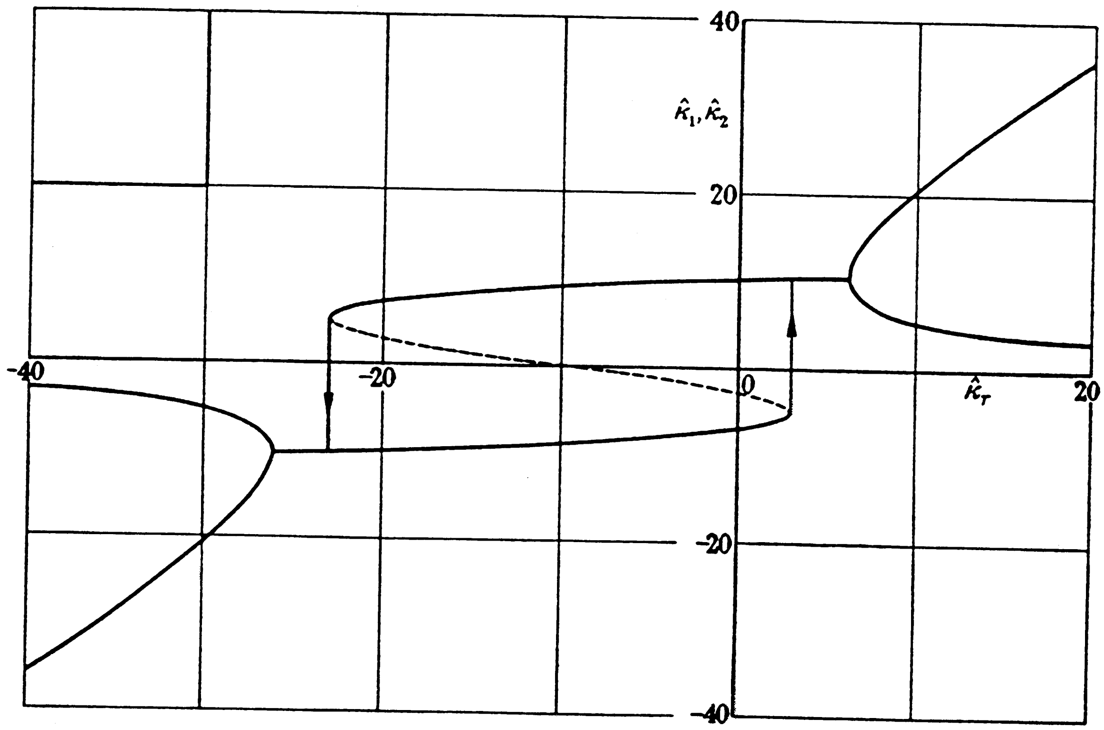


Fig. 35 Variation of Principal Curvatures with the Temperature Gradient through the Thickness ($\hat{k}_0 = 10$). [42]

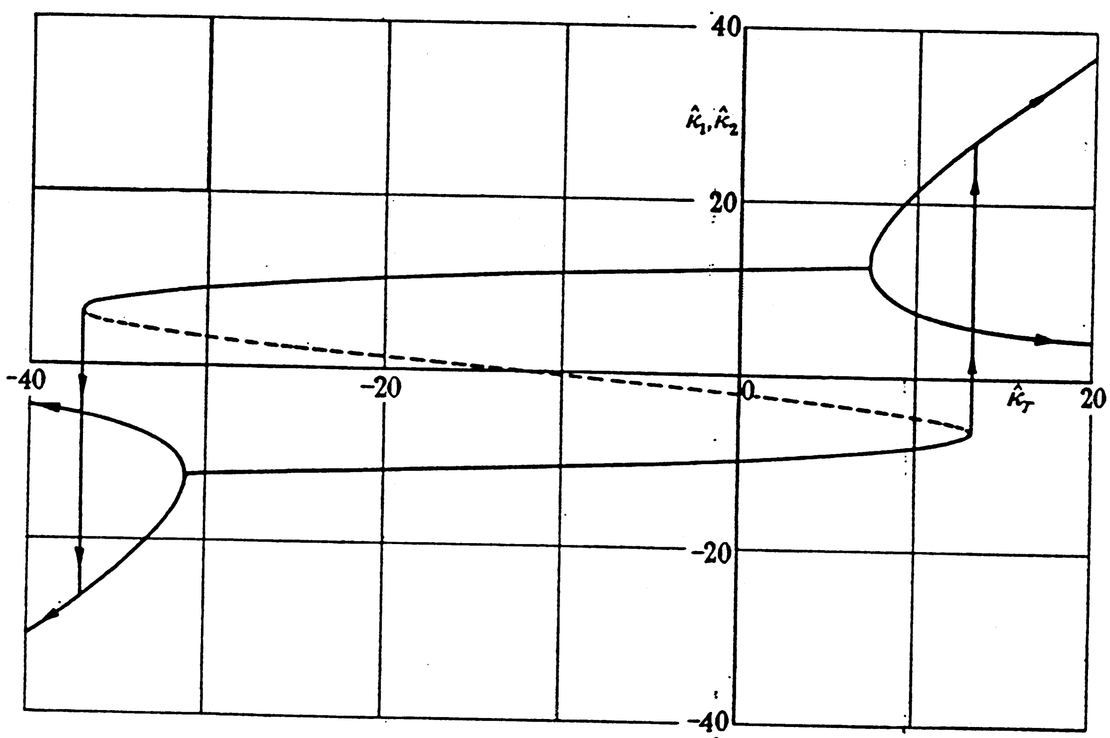


Fig. 36 Variation of Principal Curvatures with the Temperature Gradient through the Thickness ($\hat{k}_0 = 12$). [42]

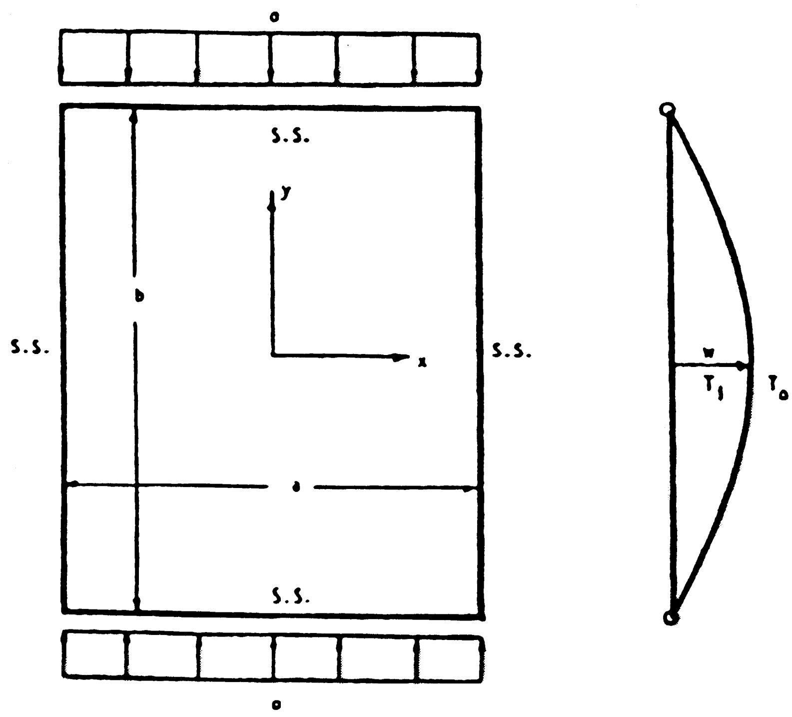


Fig. 37 Plate Dimensions, Applied Stresses, and Deflections [49]

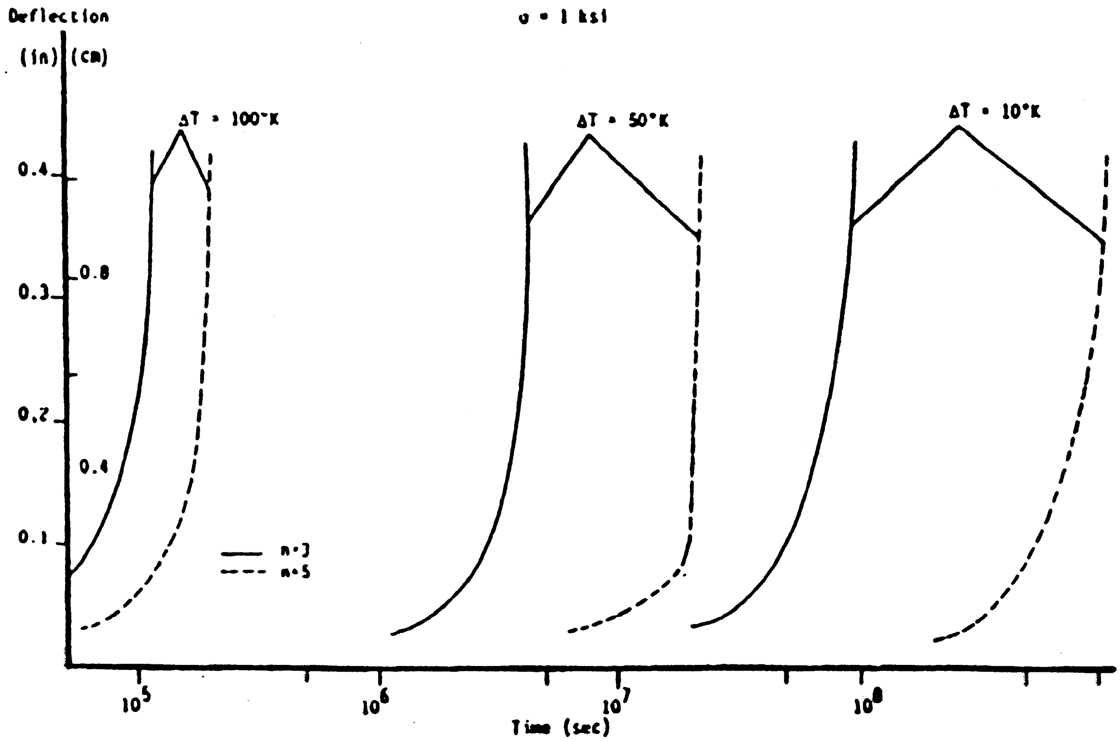


Fig. 38 Creep Deflections as a Function of Temperature Differential and Creep Exponent for an Applied Stress of 6.9 $MPa(1ksi)$. [49]

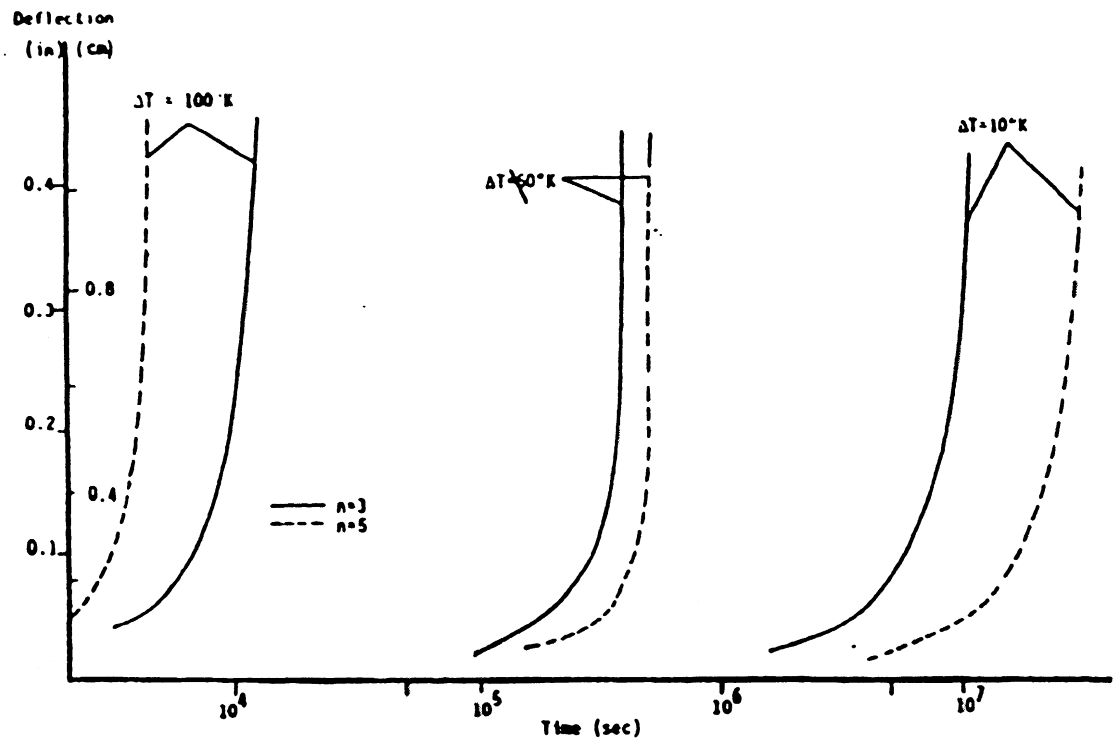


Fig. 39 Creep Deflections as a Function of Temperature Differential and Creep Exponent for an Applied Stress of 13.8 $MPa(2ksi)$. [49]

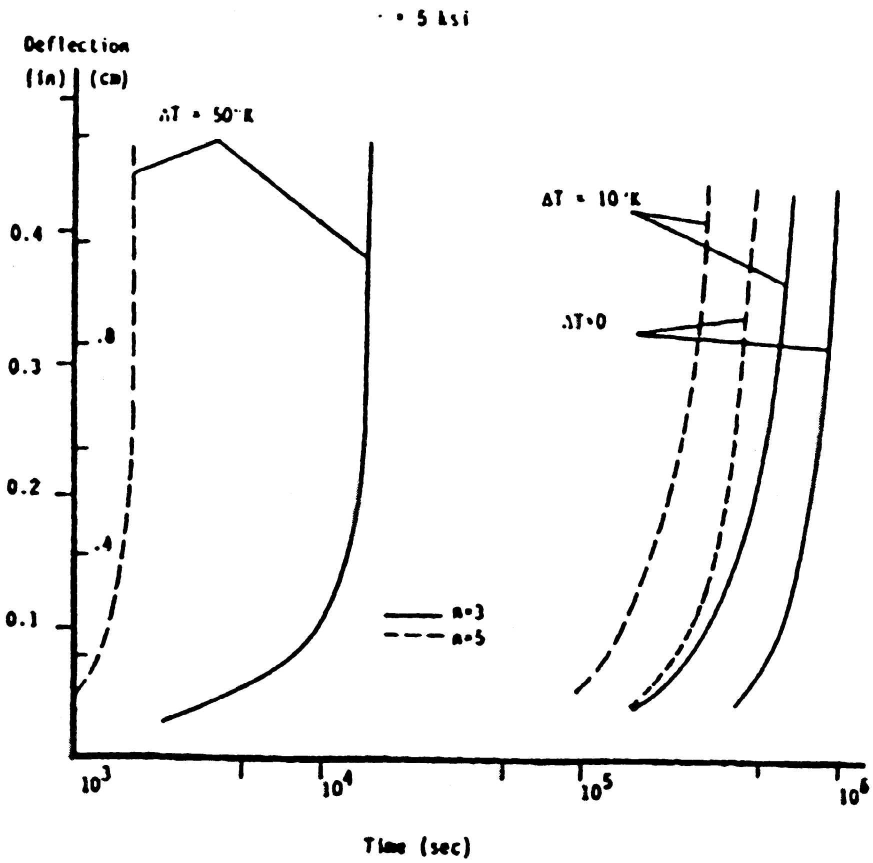


Fig. 40 Creep Deflections as a Function of Temperature Differential and Creep Exponent for an Applied Stress of 34.5 $MPa(5ksi)$. [49]

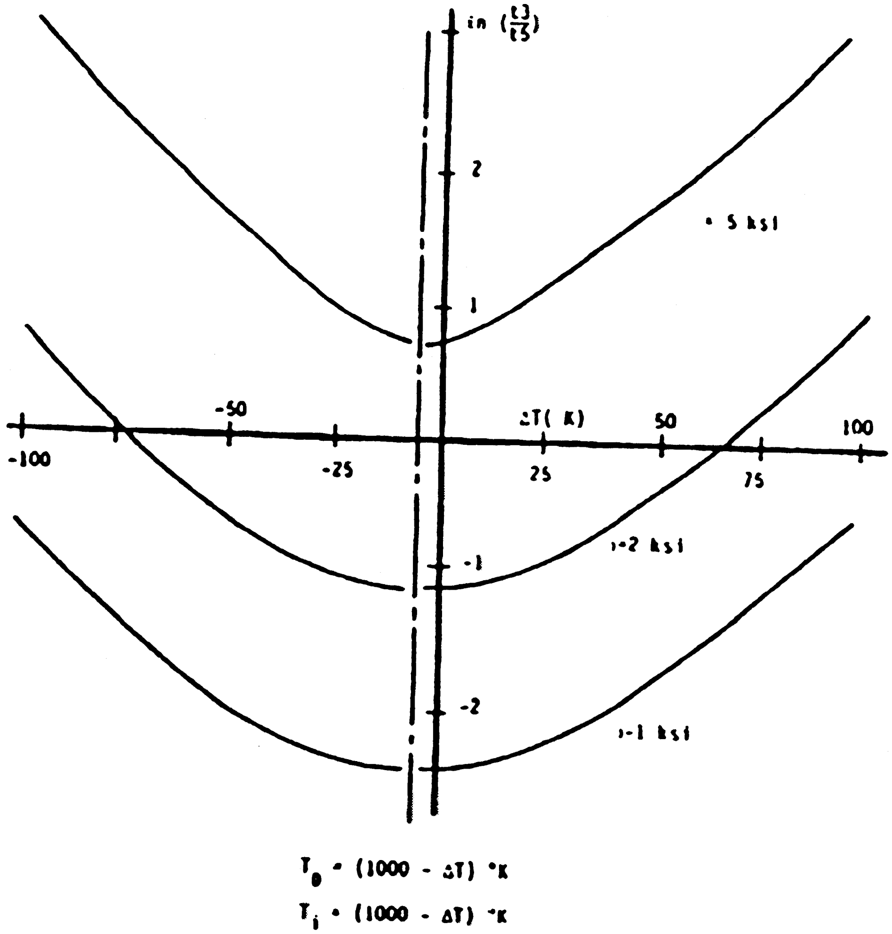


Fig. 41 Comparison of Predictions for the Time until the Creep Deflection Reaches a Specified Value for Creep Exponents $n = 3$ and $n = 5$. [49]

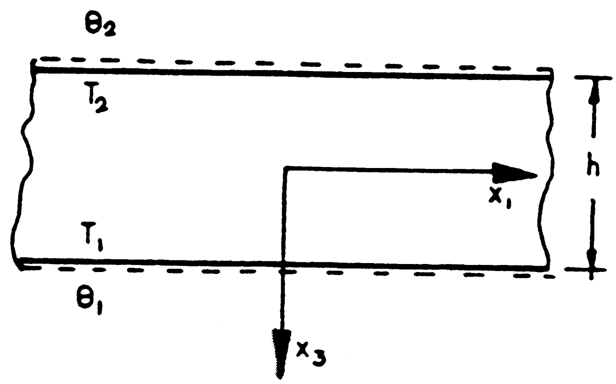


Fig. 42 The Viscoelastic Plate [50]

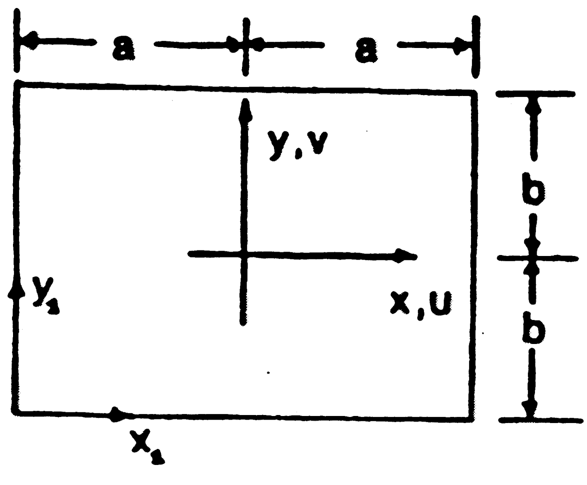


Fig. 43 Coordinate System for the Elastic-Plastic Plate [51]

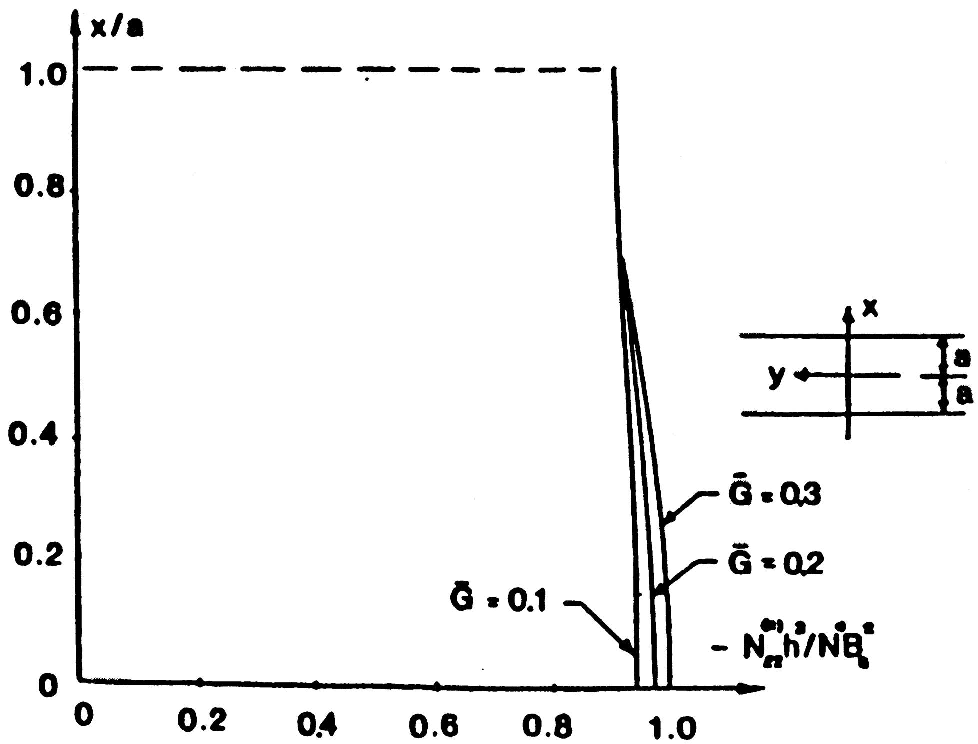


Fig. 44 Stress Factor for Simply Supported Conditions [51]

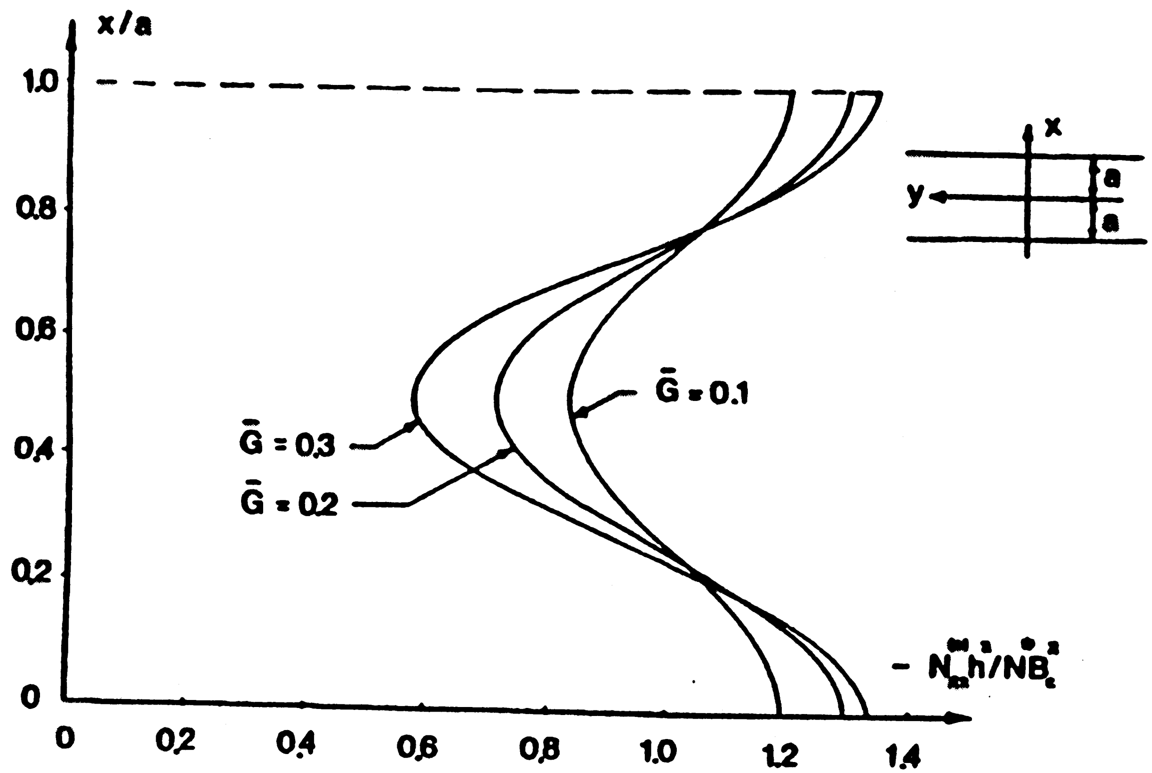


Fig. 45 Stress Factor for Clamped Conditions [51]

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